New Transformation Equations and the Electric Field Four-vector

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November 1, 2003

Abstract

In special relativity, spacetime can be described as Minkowskian. We intend to show that spacetime, as well as the laws of electromagnetism, can be described using a four-dimensional Euclidean metric as a foundation. In order to formulate these laws successfully, however, it is necessary to extend the laws of electromagnetism by replacing the Maxwell tensor with an electric field four-vector. In addition, to assure the covariance of the new laws, we introduce equations that, completely, replace the Lorentz transformation equations and Lorentz group. The above replacements, we believe, lead naturally to a unification of the electromagnetic field with the gravitational and nuclear fields. We introduce, also, a new mathematical formalism which facilitates the presentation of our laws.

1 Introduction

Lorentz first derived his famous set of transformation equations from the electromagnetic field equations of Maxwell. They assure that Maxwell's equations will have the same form in any inertial frame of reference. Unfortunately, if Maxwell's equations are shown to be incomplete, then it is likely that the Lorentz equations are incorrect. We intend to show that this is the case.

Maxwell's equations are, essentially, a set of *three-dimensional* partial differential equations. That is, each equation contains the partial derivatives with respect to only *three* of the coordinates. In four-dimensional spacetime, a three-dimensional description of anything is inherently incomplete. We will extend Maxwell's equations so that they form a set of four-dimensional equations. In so doing, it is possible to encompass *all* of Maxwell's equations in a single vector equation by introducing an electric field four-vector. In addition to the electromagnetic field, we believe the new equation incorporates the gravitational and nuclear fields. This equation, however, is *not* Lorentz invariant and requires a new set of transformation equations in order that it has the same form in all inertial frames.

The Lorentz transformation equations forbid any contraction or expansion of coordinates transverse to the direction of motion. We present a new Euclidean set of transformation equations which *require* a rotation of the coordinates transverse to the direction of motion. There is also an analogous rotation in the plane described by the direction of motion and the time coordinate.

Due to the dependence of each of the Lorentz force equations on only three of the components of the four-velocity, they also form an incomplete set of equations. Therefore, we extend these equations to four-dimensions, as well. The equation of motion then follows naturally from our new force equation.

In expressing the force equations in terms of the fields, we arrive at an energy-momentum tensor with components which include the time component of our electric field four-vector. These components offer, among other things, a new description of the mechanism behind the flow of field energy.

A new mathematical formalism is introduced which substantially simplifies the expression of our laws and helps give a deeper understanding of the geometry behind them. This new formalism borrows its structure, in part, from Hamilton's quaternions and the Clifford algebras, but differs fundamentally from both.

The form and terminology of many of the equations in this paper are, deceptively, similar to those of conventional physics, however, they differ in several ways.

NOTE: *It is important that one not assume the equivalence of the definitions presented here with the analogous definitions in conventional theory. In most cases, they are not exactly the same.*

2 Spacetime

In special relativity, spacetime can be described as Minkowskian. In this paper, we replace many of the laws of relativity and electromagnetism by using Euclidean spacetime as a foundation.

2.1 Events in Spacetime

We begin by introducing the concept of *events* in spacetime. These are the analogs in four-dimensional spacetime of points in three-dimensional space. An event is something that occurs at a specific place and at a specific time in a particular reference frame. We represent an event P in spacetime by $P(x, y, z, t)$, where x, y, z, and t are the coordinates of the event.

Events are measured by *observers* at rest in a particular reference frame who are present at the time and place of a specific event. Each observer has measured his distance from the origin of his frame by standard methods and carries a clock which has been synchronized with all other clocks in his frame by standard methods.

2.2 The Spacetime Interval

If we have two events $P_1(x_1, y_1, z_1, t_1)$ and $P_2(x_2, y_2, z_2, t_2)$, or $P(1)$ and $P(2)$ for short, in a reference frame, the magnitude of the spacetime separation between the two events $P(1)$ and $P(2)$ is called the *spacetime interval* s_{12}, which is defined as

$$s_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + c^2(t_2 - t_1)^2} \tag{2.1}$$

where we have multiplied t_1 and t_2 by the invariant speed of light in vacuo c, to make the units consistent throughout. This is, simply, the extension of the Pythagorean theorem to four dimensions, with time being the fourth dimension. To simplify, we will sometimes drop the subscripts and refer to the spacetime interval as s. In terms of the square of the *element* of spacetime interval ds^2, (2.1) becomes

$$ds^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \tag{2.2}$$

An observer in a second reference frame might measure the two events $P(1)$ and $P(2)$ to be at $P_1{}'(x_1{}', y_1{}', z_1{}', t_1{}')$ and $P_2{}'(x_2{}', y_2{}', z_2{}', t_2{}')$ or $P'(1)$ and $P'(2)$, respectively, in his frame. Notice, that we have used primes $(')$ above the coordinates, here, in order to distinguish the two sets of events in the two reference frames. In comparing quantities, we will frequently refer to the primed and unprimed quantities or frames of reference. The primed quantity will be indicated by a prime above the quantity (for example, $x_1{}'$) and the unprimed quantity by the absence of a prime above the quantity (for example, x_1). Observers in the primed frame would measure a spacetime interval

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 + c^2 dt'^2 \tag{2.3}$$

Both sets of observers, though they might measure the coordinates of the two events to be different, will agree on the spacetime interval between the events. Therefore, we can say that $ds' = ds$ or

$$dx'^2 + dy'^2 + dz'^2 + c^2 dt'^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \tag{2.4}$$

To simplify, from now on, we will take $P_1(0,0,0,0)$ and $P_1{}'(0,0,0,0)$, that is, the origins of the unprimed and primed frames coincide at $t = t' = 0$. We can, therefore, simply write $P(2)$ and $P'(2)$ as $P(x, y, z, ct)$ and $P'(x', y', z', t')$ or P and P', respectively.

2.3 The Spacetime Metric

We can write (2.2) in a more condensed form, by using the Einstein summation convention and the four-dimensional Euclidean *spacetime metric* $g_{\mu\nu} = \delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Kronecker delta,

$$\delta_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu \\ 0 & \text{for } \mu \neq \nu \end{cases} \tag{2.5}$$

Using (2.5), we can now write (2.2) as

$$ds^2 = \delta_{\mu\nu} dx_\mu dx_\nu \qquad (\mu, \nu = 1, 2, 3, 4) \tag{2.6}$$

where

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ct \tag{2.7}$$

It is typical in rectangular coordinates to use subscripts throughout, rather than the usual subscripts *and* superscripts, since there is no distinction between the covariant and contravariant components of tensors. As in the Einstein summation convention, summation is to be carried out over the *repeated* indices in each term. The greek subscripts μ, ν, ... will always range from 1 to 4, with 4 indicating time, unless otherwise noted. The latin subscripts i, j, k, ... will range from 1 to 3 and will be used to indicate spatial components, only.

3 Proper Values

If the element of spacetime interval ds is between events that are separated solely by a time interval dt (the events occur at the same place) in the unprimed frame, that is, $dx = dy = dz = 0$, then from (2.2), we have $ds = cdt$. In this case dt is defined, in this paper, as the element of *proper time* $d\tau$, so that $ds = cd\tau$. If the situation is reversed, and $dx' = dy' = dz' = 0$ in the primed frame, we can write (2.4) as

$$c^2 d\tau^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \tag{3.1}$$

Similarly, if the events are separated solely by a space interval (the events occur at the same time) in the unprimed frame, that is $dt = 0$, then from (2.2), $ds = \sqrt{dx^2 + dy^2 + dz^2}$. In this case, $\sqrt{dx^2 + dy^2 + dz^2}$ is defined as the element of *proper length* $d\lambda$, therefore (2.2) becomes $ds = d\lambda$. If, on the other hand, $dt' = 0$ in the primed frame, we can write (2.4) as

$$d\lambda^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \tag{3.2}$$

We put no primes on the proper time or length, since they are the same in all reference frames.

The proper time τ and the proper length λ are always the *maximum* possible measurements of time and length made between events in any frame and can be measured, directly, only by inertial observers. If there are space *and* time components of the spacetime interval between events in a reference frame, then *neither* component is proper and both will be less than the proper value. Therefore, we refer to them as *improper* values. In general, the proper value of *any* quantity, in this paper, will be its maximum value.

4 Four-vectors

We will represent an arbitrary *four-vector* $\mathbf{A}$ by two equivalent methods. The first method, describing $\mathbf{A}$ in terms of the basis vectors $\mathbf{e}_\mu$ is

$$\mathbf{A} = A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3 + A_4\,\mathbf{e}_4 \tag{4.1}$$

where

$$A_1 = A_x, \qquad A_2 = A_y, \qquad A_3 = A_z, \qquad A_4 = A_t \tag{4.2}$$

or equivalently $\mathbf{A} = A_\mu\,\mathbf{e}_\mu$, where the A_μ are the components of the four-vector $\mathbf{A}$ in the directions of the basis vectors $\mathbf{e}_\mu$. The second method we will often use is

$$A_\mu = (A_1, A_2, A_3, A_4) \tag{4.3}$$

which is, simply, another way to express (4.1). Both of these methods will be used to represent four-vectors.

The *norm* or magnitude of our arbitrary four-vector $|\mathbf{A}|$ is defined as the invariant

$$|\mathbf{A}| = \sqrt{\delta_{\mu\nu} A_\mu A_\nu} \tag{4.4}$$

All vectors will be represented in boldface type, with four-vectors indicated in uppercase type and three-vectors in lowercase type, unless otherwise noted. Basis vectors will also be indicated in bold lowercase type, but will be accompanied by a subscript.

4.1 The Multiplication of Four-vectors

We introduce now what we believe is a new mathematical formalism which we will use to simplify and condense our laws. These rules have been derived, in part, from Hamilton's quaternions, and the Clifford algebras.

4.1.1 Basis Vectors

It is possible to multiply four-vectors algebraically by using the appropriate conventions for the products of the orthonormal *basis vectors* $\mathbf{e}_\mu$ of a particular reference frame. The basis vectors $\mathbf{e}_\mu$ must satisfy

$$\mathbf{e}_\mu \mathbf{e}_\nu = -\mathbf{e}_\nu \mathbf{e}_\mu \qquad \text{for } \mu \neq \nu \textit{ and } \text{for } \mu = \nu \tag{4.5}$$

for $\mu, \nu = 1, 2, 3, 4$. The basis vectors $\mathbf{e}_\mu$ must, also, satisfy either the relations

$$\begin{aligned} &\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i = \epsilon_{ijk} \mathbf{e}_k \qquad \text{for } i \neq j \\ &\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_4 \mathbf{e}_4 = \mathbf{e}_4 \qquad \text{for } i = j \\ &\mathbf{e}_i \mathbf{e}_4 = -\mathbf{e}_4 \mathbf{e}_i = \mathbf{e}_i \end{aligned} \tag{4.6}$$

or the relations

$$\begin{aligned} &\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i = \epsilon_{ijk} \mathbf{e}_k \qquad \text{for } i \neq j \\ &\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_4 \mathbf{e}_4 = \mathbf{e}_4 \qquad \text{for } i = j \\ &\mathbf{e}_i \mathbf{e}_4 = \mathbf{e}_4 \mathbf{e}_i = \mathbf{e}_i \end{aligned} \tag{4.7}$$

for $i, j, k = 1, 2, 3$, where ϵ_{ijk} is the three-dimensional permutation symbol. The basis vectors on the right-hand sides of the set of rules (4.6) can be either positive or negative, independently of each other, that is, we could have written (4.6) as

$$\begin{aligned} &\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i = \epsilon_{ijk} \mathbf{e}_k \qquad \text{for } i \neq j \\ &\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_4 \mathbf{e}_4 = -\mathbf{e}_4 \qquad \text{for } i = j \\ &\mathbf{e}_i \mathbf{e}_4 = -\mathbf{e}_4 \mathbf{e}_i = \mathbf{e}_i \end{aligned} \tag{4.8}$$

or, alternately,

$$\begin{aligned} &\mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i = \epsilon_{ijk} \mathbf{e}_k \qquad \text{for } i \neq j \\ &\mathbf{e}_i \mathbf{e}_j = \mathbf{e}_4 \mathbf{e}_4 = -\mathbf{e}_4 \qquad \text{for } i = j \\ &\mathbf{e}_i \mathbf{e}_4 = -\mathbf{e}_4 \mathbf{e}_i = -\mathbf{e}_i \end{aligned} \tag{4.9}$$

and so on. Similarly for the set of rules (4.7). Considering all possible combinations, we obtain eight possible sets of rules for (4.6), and additionally, eight possible sets of rules for (4.7), resulting in a total of sixteen possible sets of rules. As we will see later, these sixteen sets of rules can also be combined to obtain additional results. The basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are, generally, to be regarded as spatial, and the basis vector $\mathbf{e}_4$ is to be regarded as temporal and is directed along the worldline or timeline of the reference frame.

In general, the rules (4.6) are non-associative as well as non-commutative, for example using (4.6),

$$(\mathbf{e}_1 \mathbf{e}_4)\mathbf{e}_3 \neq \mathbf{e}_1(\mathbf{e}_4 \mathbf{e}_3) \tag{4.10}$$

We believe that this property of non-associativity has important physical significance, just as the property of non-commutativity has been shown to have important physical significance.

4.1.2 The Four-vector Product

Using these rules, we can write the product of two four-vectors as a normal algebraic product. For example, the *four-vector product* $\mathbf{AB}$ of two arbitrary four-vectors $\mathbf{A} = A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3 + A_4\,\mathbf{e}_4$ and $\mathbf{B} = B_1\,\mathbf{e}_1 + B_2\,\mathbf{e}_2 + B_3\,\mathbf{e}_3 + B_4\,\mathbf{e}_4$ is

$$\mathbf{AB} = (A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3 + A_4\,\mathbf{e}_4)(B_1\,\mathbf{e}_1 + B_2\,\mathbf{e}_2 + B_3\,\mathbf{e}_3 + B_4\,\mathbf{e}_4) \tag{4.11}$$

We will choose one of the eight possible sets of the rules for the products of basis vectors for our first example, but as will be shown later, multiple sets may be used in a given product. Let us choose, for this example, the rules in (4.6). Multiplying (4.11) algebraically, using (4.6) for the products of the basis vectors, we get

$$\begin{aligned}\mathbf{AB} = {}& (A_2B_3 - A_3B_2 + A_1B_4 - A_4B_1)\,\mathbf{e}_1 \\ &+ (A_3B_1 - A_1B_3 + A_2B_4 - A_4B_2)\,\mathbf{e}_2 \\ &+ (A_1B_2 - A_2B_1 + A_3B_4 - A_4B_3)\,\mathbf{e}_3 \\ &+ (A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4)\,\mathbf{e}_4\end{aligned} \tag{4.12}$$

Consequently, the product of two four-vectors results in another four-vector. Note that we could have just as easily used (4.7) or any other of the possible sets of rules, rather than (4.6), for the product to obtain a different, but equally valid, result.

If we substitute the four-vector $\mathbf{A}$ in place of the four-vector $\mathbf{B}$ in (4.12), we get

$$\begin{aligned}\mathbf{AA} = {}& (A_2A_3 - A_3A_2 + A_1A_4 - A_4A_1)\,\mathbf{e}_1 \\ &+ (A_3A_1 - A_1A_3 + A_2A_4 - A_4A_2)\,\mathbf{e}_2 \\ &+ (A_1A_2 - A_2A_1 + A_3A_4 - A_4A_3)\,\mathbf{e}_3 \\ &+ (A_1A_1 + A_2A_2 + A_3A_3 + A_4A_4)\,\mathbf{e}_4\end{aligned} \tag{4.13}$$

Notice that the spatial components vanish, and we are left with

$$\mathbf{AA} = (A_1A_1 + A_2A_2 + A_3A_3 + A_4A_4)\,\mathbf{e}_4 \tag{4.14}$$

Interestingly, the magnitude of the time component of $\mathbf{AA}$ (or $\mathbf{A}^2$), in this case, is identical to the magnitude of $\mathbf{AA}$, as well as the square of the magnitude of the four-vector $\mathbf{A}$ from (4.4), since

$$|\mathbf{A}^2| = A_1A_1 + A_2A_2 + A_3A_3 + A_4A_4 = \delta_{\mu\nu}A_\mu A_\nu = |\mathbf{A}|^2 \tag{4.15}$$

4.1.3 The Derivative Product

Using the same methods as above, we can write the *derivative product* of a four-vector (or derivative of a four-vector, for short) in vector form. For example, the derivative product of our arbitrary four-vector $\mathbf{A} = A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3 + A_4\,\mathbf{e}_4$ and the *derivative four-vector*, $\mathbf{d} = \mathbf{e}_1\,\partial_1 + \mathbf{e}_2\,\partial_2 + \mathbf{e}_3\,\partial_3 + \mathbf{e}_4\,\partial_4$, where

$$\partial_1 = \frac{\partial}{\partial x_1}, \qquad \partial_2 = \frac{\partial}{\partial x_2}, \qquad \partial_3 = \frac{\partial}{\partial x_3}, \qquad \partial_4 = \frac{\partial}{\partial x_4} \tag{4.16}$$

can be written,

$$\mathbf{dA} = (\mathbf{e}_1\,\partial_1 + \mathbf{e}_2\,\partial_2 + \mathbf{e}_3\,\partial_3 + \mathbf{e}_4\,\partial_4)(A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3 + A_4\,\mathbf{e}_4) \tag{4.17}$$

Multiplying, algebraically using the rules (4.6), as before, we get

$$\begin{aligned}\mathbf{dA} &= (\partial_2 A_3 - \partial_3 A_2 + \partial_1 A_4 - \partial_4 A_1)\,\mathbf{e}_1 \\ &+ (\partial_3 A_1 - \partial_1 A_3 + \partial_2 A_4 - \partial_4 A_2)\,\mathbf{e}_2 \\ &+ (\partial_1 A_2 - \partial_2 A_1 + \partial_3 A_4 - \partial_4 A_3)\,\mathbf{e}_3 \\ &+ (\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 + \partial_4 A_4)\,\mathbf{e}_4\end{aligned} \tag{4.18}$$

where

$$\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x_\mu} \tag{4.19}$$

To differentiate a vector product, say the product of our two arbitrary four-vectors $\mathbf{A}$ and $\mathbf{B}$, we use the usual product rule

$$\mathbf{d}((\mathbf{AB})) = (\mathbf{dA})\mathbf{B} + \mathbf{A}(\mathbf{dB}) \tag{4.20}$$

Single parentheses surrounding a pair of four-vectors, in a triple product, indicate that we are to multiply the four-vectors in parentheses *before* multiplying by the four-vector outside the parentheses. On the other hand, double parentheses indicate that we are *not* to multiply the four-vectors in parentheses first. Without this distinction, $\mathbf{d}((\mathbf{AB}))$ and $\mathbf{d}(\mathbf{AB})$ might be misinterpreted as being equivalent.

Similarly, the second derivative of an arbitrary four-vector takes the form

$$\mathbf{d}((\mathbf{dA})) = (\mathbf{dd})\mathbf{A} + \mathbf{d}(\mathbf{dA}) \tag{4.21}$$

On the right-hand side of (4.21), note that we have used single parentheses to indicate that the multiplications $(\mathbf{dd})$ and $(\mathbf{dA})$ are to be carried out first.

It can be shown easily by multiplying the derivative four-vector by itself that one possible result is

$$\mathbf{dd} = \mathbf{d}^2 = \partial^2 \tag{4.22}$$

where

$$\mathbf{d}^2 = \partial^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{c^2 \partial t^2} \tag{4.23}$$

After carrying out the multiplication on the right-hand side of (4.21) we find that, in general,

$$\mathbf{d}((\mathbf{dA})) = 0 \tag{4.24}$$

4.1.4 Combined Products

As mentioned, previously, it is possible to combine sets of rules for the products of basis vectors in a single four-vector product. For example, we could combine (4.6) and (4.7), to get a combination of the two sets of rules. The resulting signs of the terms are a *superposition* of the signs of *both* sets of rules. Using both sets of rules, (4.6) and (4.7), in (4.11), we have

$$\begin{aligned} \mathbf{AB} &= (A_2B_3 - A_3B_2 + A_1B_4 \mp A_4B_1)\,\mathbf{e}_1 \\ &+ (A_3B_1 - A_1B_3 + A_2B_4 \mp A_4B_2)\,\mathbf{e}_2 \\ &+ (A_1B_2 - A_2B_1 + A_3B_4 \mp A_4B_3)\,\mathbf{e}_3 \\ &+ (A_1B_1 + A_2B_2 + A_3B_3 \pm A_4B_4)\,\mathbf{e}_4 \end{aligned} \tag{4.25}$$

This is only one of the *combined products* possible. Other products can be created by combining any of the sixteen sets of rules in the manner of (4.25).

We will frequently use component notation along with vector notation in our descriptions. However, it is impossible to include all possible combinations of components contained in a given vector equation, in a single component equation. Therefore, any equation expressed in component form should, in general, be considered as only one possible form of the vector equation from which it was derived.

If we refer to the signs, "$\pm$" and "$\mp$", in (4.25) as "plus and minus" and "minus and plus", respectively, then the signs preceding the A_2B_3, A_3B_1, and A_1B_2 terms are, actually, "plus and plus", and the signs preceding the A_3B_2, A_1B_3, and A_2B_1 terms are "minus and minus". Unfortunately, there are no displayable mathematical symbols of this kind available.

In the case of terms preceded by "$\pm$" or "$\mp$", the signs retain their opposite nature, even though each sign contains both "+" and "−" signs. For convenience, the terms preceded by "plus and plus" and "minus and minus" can be considered as "+" and "-", respectively. The signs "$\pm$", "$\mp$", "plus and plus", and "minus and minus", will be referred to as *combined signs*. The upper sign in the combination will always represent a single set of rules, throughout, and the lower sign will represent a single set of rules, throughout. It is important to note, however, that these combined signs are *not* to be mistaken as "plus *or* minus", "minus *or* plus", "plus *or* plus" or "minus *or* minus".

4.1.5 Vector Notation

The four-vector product **AB** can be written more compactly, using *vector notation*, as

$$\mathbf{AB} = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B} + \mathbf{A}:\mathbf{B} \tag{4.26}$$

where, in the case of (4.12),

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= (A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4)\,\mathbf{e}_4 \\ \mathbf{A}\times\mathbf{B} &= (A_2B_3 - A_3B_2)\,\mathbf{e}_1 + (A_3B_1 - A_1B_3)\,\mathbf{e}_2 + (A_1B_2 - A_2B_1)\,\mathbf{e}_3 \\ \mathbf{A}:\mathbf{B} &= (A_1B_4 - A_4B_1)\,\mathbf{e}_1 + (A_2B_4 - A_4B_2)\,\mathbf{e}_2 + (A_3B_4 - A_4B_3)\,\mathbf{e}_3 \end{aligned} \tag{4.27}$$

The signs of the terms on the right-hand sides of (4.27) reflect the product rules (4.6), in this case, but can also represent any of the sixteen product rules. They can also represent combined products, by using combined signs, rather than single signs.

Just as in four-vector product, the derivative product **dA** can be written in condensed form as

$$\mathbf{dA} = \mathbf{d}\cdot\mathbf{A} + \mathbf{d}\times\mathbf{A} + \mathbf{d}:\mathbf{A} \tag{4.28}$$

where, in the case of (4.18),

$$\begin{aligned}\mathbf{d}\cdot\mathbf{A} &= (\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 + \partial_4 A_4)\,\mathbf{e}_4 \\ \mathbf{d}\times\mathbf{A} &= (\partial_2 A_3 - \partial_3 A_2)\,\mathbf{e}_1 + (\partial_3 A_1 - \partial_1 A_3)\,\mathbf{e}_2 + (\partial_1 A_2 - \partial_2 A_1)\,\mathbf{e}_3 \\ \mathbf{d} : \mathbf{A} &= (\partial_1 A_4 - \partial_4 A_1)\,\mathbf{e}_1 + (\partial_2 A_4 - \partial_4 A_2)\,\mathbf{e}_2 + (\partial_3 A_4 - \partial_4 A_3)\,\mathbf{e}_3\end{aligned} \tag{4.29}$$

The product $\mathbf{d}\cdot\mathbf{A}$ is the four-divergence of $\mathbf{A}$, $\mathbf{d}\times\mathbf{A}$ is the curl of $\mathbf{A}$, and $\mathbf{d} : \mathbf{A}$ is a new product we will call the *evolution* of $\mathbf{A}$. Again, the signs of the terms on the right-hand sides of (4.29) represent (4.18), in this case, but can be changed to represent any of the sixteen product rules or combined products. We can, therefore, easily represent any of the possible results for four-vector products, derivative products, or combined products, in vector notation.

5 The Four-velocity

Suppose that the primed frame of reference is in uniform motion with respect to the unprimed frame. Reference frames at rest or in uniform motion with respect to each other are referred to as *inertial* reference frames. This motion is represented in four-dimensional spacetime by the velocity four-vector, or *four-velocity* $\mathbf{U}$.

The components of the four-velocity U_μ of the primed frame, according to an observer at rest in the unprimed frame (unprimed observer), are

$$U_\mu = \frac{dx_\mu}{d\tau} \tag{5.1}$$

where we have used the proper time τ in the denominator rather than the coordinate time t in order that the components U_μ form a four-vector. We represent the velocity four-vector by $\mathbf{U} = U_\mu\,\mathbf{e}_\mu$ or, equivalently, by $U_\mu = (U_1, U_2, U_3, U_4)$. In the future we will assume, in general, that the unprimed frame of reference is at rest, and that the primed frame of reference is in uniform motion with respect to the unprimed frame. As we will show next, the magnitude of the four-velocity is invariant.

5.1 Invariant Magnitude of the Four-velocity

Let us imagine that two events occur at the same place, but at different times in the primed reference frame, that is, $dx' = dy' = dz' = 0$. Since an observer at rest in the primed frame (primed observer) measures no space interval between the events, his measurement of the spacetime interval is entirely temporal. Therefore, as discussed in Section 3,

$$ds' = cd\tau \tag{5.2}$$

Therefore, from (3.1), we can write

$$c^2 d\tau^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \tag{5.3}$$

Dividing both sides of (5.3) by $d\tau^2$, we get

$$c^2 = \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 + \left(\frac{cdt}{d\tau}\right)^2 \tag{5.4}$$

or, from (5.1) and (5.4),

$$c^2 = U_x^2 + U_y^2 + U_z^2 + U_t^2 \tag{5.5}$$

Now, the norm or magnitude of the four-velocity $|\mathbf{U}|$, using (4.4), is defined as

$$|\mathbf{U}| = \sqrt{\delta_{\mu\nu} U_\mu U_\nu} \tag{5.6}$$

but

$$\sqrt{\delta_{\mu\nu} U_\mu U_\nu} = \sqrt{U_x^2 + U_y^2 + U_z^2 + U_t^2} \tag{5.7}$$

so, from (5.5), (5.6), and (5.7),

$$|\mathbf{U}| = c \tag{5.8}$$

Since $\mathbf{U}$ represents an arbitrary four-velocity, we conclude that the magnitude of the four-velocity of *any* body is the *invariant* speed of light, c. In the case of a body at *rest*, the four-velocity is $U_\mu = (0, 0, 0, c)$, where $U_t = c$.

6 The Transformation Equations

We wish to find a set of coordinate transformation equations that assure the covariance of the laws of physics described in this paper. Initially, we are making a transformation of coordinates from a stationary unprimed frame of reference to a uniformly moving primed frame, so we assume that the transformation involves the four-velocity $U_\mu = (U_x, U_y, U_z, U_t)$ of the moving frame. But the transformed coordinates must have the same units as the original coordinates, therefore, we divide the U_μ by the invariant magnitude of the four-velocity, c.

We take the origins of the unprimed and primed frames to coincide at $t = t' = 0$ and the x, y, z, and t axes to be parallel to the corresponding axes of the primed frame when both frames are at rest. Let us define the *position four-vector* $\mathbf{X}$ in the unprimed frame, as

$$\mathbf{X} = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3 + ct\,\mathbf{e}_4 \tag{6.1}$$

which is directed from the origin of the unprimed frame to an arbitrary event $P(x, y, z, t)$ in the unprimed frame, and the position four-vector $\mathbf{X}'$ in the primed frame, as

$$\mathbf{X}' = x'\,\mathbf{e}_1 + y'\,\mathbf{e}_2 + z'\,\mathbf{e}_3 + ct'\,\mathbf{e}_4 \tag{6.2}$$

directed from the origin of the primed frame to the same event $P'(x', y', z', t')$ in the primed frame. A coordinate transformation is, essentially, the operation of transforming the four-vector $\mathbf{X}$ into the four-vector $\mathbf{X}'$. This is accomplished through the four-vector product $(1/c)\,\mathbf{UX}$ or *transformation equation*,

$$\mathbf{X}' = \frac{1}{c}\,\mathbf{UX} \tag{6.3}$$

We can expand the right-hand side of (6.3), using one of the possible combined products, to get

$$\begin{aligned} \mathbf{X}' = (1/c)\,((&\pm U_t x \pm U_z y \mp U_y z \mp U_x ct)\,\mathbf{e}_1 \\ +(&\mp U_z x \pm U_t y \pm U_x z \mp U_y ct)\,\mathbf{e}_2 \\ +(&\pm U_y x \mp U_x y \pm U_t z \mp U_z ct)\,\mathbf{e}_3 \\ +(&\pm U_x x \pm U_y y \pm U_z z \pm U_t ct)\,\mathbf{e}_4) \end{aligned} \tag{6.4}$$

Equating components from the right-hand sides of (6.2) and (6.4), we find

$$\begin{aligned} x' &= \frac{1}{c}\left(\pm U_t x \pm U_z y \mp U_y z \mp U_x ct\right) \\ y' &= \frac{1}{c}\left(\mp U_z x \pm U_t y \pm U_x z \mp U_y ct\right) \\ z' &= \frac{1}{c}\left(\pm U_y x \mp U_x y \pm U_t z \mp U_z ct\right) \\ ct' &= \frac{1}{c}\left(\pm U_x x \pm U_y y \pm U_z z \pm U_t ct\right) \end{aligned} \tag{6.5}$$

or, in condensed form, (6.5) becomes

$$x_\mu{}' = U_{\mu\nu} x_\nu \tag{6.6}$$

where

$$U_{\mu\nu} = \frac{1}{c}\begin{pmatrix} \pm U_t & \pm U_z & \mp U_y & \mp U_x \\ \mp U_z & \pm U_t & \pm U_x & \mp U_y \\ \pm U_y & \mp U_x & \pm U_t & \mp U_z \\ \pm U_x & \pm U_y & \pm U_z & \pm U_t \end{pmatrix} \tag{6.7}$$

The matrix $U_{\mu\nu}$ in (6.7) (not to be confused with the velocity four-vector $\mathbf{U}$ with components U_μ) will be referred to as a *transformation matrix*. Of course, there are other possible choices for the components of $U_{\mu\nu}$ resulting from the use of alternate combined products of basis vectors. For example, without altering the signs of the other terms, we could have reversed the signs of U_{12}, U_{13}, U_{21}, U_{23}, U_{31}, and U_{32} in (6.7), while preserving the orthogonality of $U_{\mu\nu}$, as required in Euclidean spacetime. The equation (6.3) should be seen to represent any of the possible combined products which leave the Euclidean spacetime interval (2.1) invariant.

We can get a feeling for the geometrical meaning of (6.3) by giving a simplified example. Imagine that the primed frame is in uniform motion with four-velocity $U_\mu = (U_x, 0, 0, U_t)$ relative to the unprimed frame. To simplify, we will consider only the positive U_t components in (6.7), here, although the negative U_t components are equally significant. In this case, (6.5) becomes

$$\begin{aligned} x' &= \frac{1}{c}\left(U_t x \mp U_x ct\right) \\ y' &= \frac{1}{c}\left(U_t y \pm U_x z\right) \\ z' &= \frac{1}{c}\left(\mp U_x y + U_t z\right) \\ ct' &= \frac{1}{c}\left(\pm U_x x + U_t ct\right) \end{aligned} \tag{6.8}$$

We see that the y'-z' plane is rotated in clockwise *and* counterclockwise directions in the y-z plane, *simultaneously*, by the angle $\theta = \arctan(U_x/U_t)$ and that the x'-t' plane is also similarly rotated in the x-t plane.

It is important to note that we are describing the components of the spacetime interval from the event $P_1(0,0,0,0)$ to the event $P_2(x,y,z,t)$, in the unprimed frame, and the components of the spacetime interval from the event $P_1'(0,0,0,0)$ to the event $P_2'(x',y',z',t')$, in the primed frame, and not simply the coordinates of the events P_2 and P_2'. But since the events P_1 and P_1'

are at the origins of the two frames, the components of the spacetime intervals, in each frame, are just the coordinates of P_2 and P_2'.

It is also important to note that the unprimed frame is considered to be the rest frame of the events, in this case. Therefore, measurements are made, in the primed frame, between the apparent positions and times of the events *in the unprimed frame.*

6.1 Inverse Transformation Equations

We can find the inverse transformation equations, that is, the equations describing the transformation of coordinates from the primed frame to the unprimed frame, by remembering that, according to a primed observer, the unprimed frame is moving in the opposite direction. Therefore, by substituting $U'_\mu = (-U_x, -U_y, -U_z, U_t)$ for the four-velocity of the unprimed frame relative to the primed frame and switching the four-vectors $\mathbf{X}$ and $\mathbf{X}'$ in (6.3) we get the *inverse transformation equation*

$$\mathbf{X} = \frac{1}{c}\,\mathbf{U}'\mathbf{X}' \tag{6.9}$$

or in component form,

$$x_\nu = x_\mu{}' U_{\mu\nu} \tag{6.10}$$

The transformation equations (6.3), (6.6), (6.9), *and* (6.10) *REPLACE the Lorentz transformation equations and the Lorentz group.*

7 Transformation of Length

Assume that the primed frame is in uniform motion with four-velocity $U_\mu = (U_x, 0, 0, U_t)$ relative to the unprimed frame. In order to compare measurements of the spatial interval between events in the direction of motion in the two frames, we take the interval between the origin and the event $P(x, 0, 0, 0)$ in the unprimed frame. We wish to find the coordinates of the same event $P'(x', y', z', t')$, in the primed frame. To transform coordinates between the unprimed and primed frames, we will use (6.6). Expanding (6.6), using U_μ and P above, we have

$$\begin{aligned} x' &= \frac{1}{c}\,(U_t x) \\ ct' &= \frac{1}{c}\,(\pm U_x x) \end{aligned} \tag{7.1}$$

Since we are comparing spatial intervals, we are interested in the first equation in (7.1),

$$x' = \frac{U_t}{c}\,x \tag{7.2}$$

The U_t/c part of (7.2) can be put in a more familiar form by remembering from (5.5) that

$$c^2 = U_x^2 + U_y^2 + U_z^2 + U_t^2 \tag{7.3}$$

After rearranging terms, we have

$$\frac{U_t}{c} = \sqrt{1 - U^2/c^2} \tag{7.4}$$

where

$$U^2 = U_x^2 + U_y^2 + U_z^2 \tag{7.5}$$

In order to simplify, in the future, we define

$$\gamma = \frac{U_t}{c} = \sqrt{1 - U^2/c^2} \tag{7.6}$$

Inserting (7.6) into (7.2) we get

$$x' = \gamma x \tag{7.7}$$

The coordinate x in (7.7), in this case, is the proper length λ, since $y = z = t = 0$, and the coordinate x' in (7.7) is the improper length.

8 Transformation of Time

We can use similar methods to compare elapsed times between events in two reference frames. Using the four-velocity $U_\mu = (U_x, 0, 0, U_t)$ from Section 7 and the interval between the origin and the event $P(0, 0, 0, t)$ in the unprimed frame, we employ (6.6), again, to find the coordinates of the same event $P'(x', y', z', t')$, in the primed frame, to get

$$\begin{aligned} x' &= \frac{1}{c}\left(\mp U_x ct\right) \\ ct' &= \frac{1}{c}\left(U_t ct\right) \end{aligned} \tag{8.1}$$

But since we are comparing time measurements, we consider the second equation in (8.1). After dividing by c, we get

$$t' = \frac{U_t}{c}\, t \tag{8.2}$$

or, inserting (7.6) into (8.2), we have

$$t' = \gamma t \tag{8.3}$$

The coordinate t in (8.3), in this case, is the proper time τ, since $x = y = z = 0$, and the coordinate t' in (8.3) is the improper time.

9 The Transformation of Velocity

If we have a body in uniform motion relative to the primed frame, and the primed frame is in uniform motion relative to the unprimed frame, an observer in the primed frame can find the bodies motion relative to the unprimed frame by using the equations for the inverse transformation of coordinates (6.9) or (6.10) with the appropriate substitutions. From this point on, in order to simplify, we will display only one of the signs from the combined sign of each term.

Imagine that the body is moving with uniform four-velocity $V'_\mu = (V'_x, V'_y, V'_z, V'_t)$ relative to the primed frame and that the unknown four-velocity of the body relative to the unprimed frame is $V_\mu = (V_x, V_y, V_z, V_t)$. The primed frame, in turn, is moving with uniform four-velocity $U_\mu =$

(U_x, U_y, U_z, U_t) relative to the unprimed frame. The primed observer can find $\mathbf{V}$ by substituting $\mathbf{V}$ and $\mathbf{V}'$ for $\mathbf{X}$ and $\mathbf{X}'$, respectively, in (6.9) to get

$$\mathbf{V} = \frac{1}{c}\,\mathbf{U}\mathbf{V}' \tag{9.1}$$

or alternately, we can make the same substitutions V_ν and $V_\mu{}'$ in place of x_ν and $x_\mu{}'$, respectively, in (6.10) to obtain the same results in component form

$$V_\nu = V_\mu{}' U_{\mu\nu} \tag{9.2}$$

For $\mathbf{U}$ and $\mathbf{V}'$ in the same direction, for example $U_\mu = (U_x, 0, 0, U_t)$ and $V'_\mu = (V'_x, 0, 0, V'_t)$ and using (9.2), we get

$$\begin{aligned} V_x &= \frac{1}{c}\left(U_t V'_x + U_x V'_t\right) \\ V_t &= \frac{1}{c}\left(-U_x V'_x + U_t V'_t\right) \end{aligned} \tag{9.3}$$

For $U_x << c$ and $V'_x << c$, we have $U_t \approx c$ and $V'_t \approx c$, so that we get, approximately, the Galilean result $V_x \approx V'_x + U_x$.

In the case of $U_\mu = (c, 0, 0, 0)$ and $V'_\mu = (c, 0, 0, 0)$ we get $V_x = V_y = V_z = 0$ and $V_t = -c$ or $V_\mu = (0, 0, 0, -c)$.

10 The Current Density Four-vector

The *current density four-vector* $\mathbf{J}$ for a distribution of charge(s) moving with four-velocity $\mathbf{U}$ is defined as

$$\mathbf{J} = \frac{\rho_e}{c}\,\mathbf{U} \tag{10.1}$$

where ρ_e is the *charge density*, or amount of charge per unit volume. The components J_μ of the current density four-vector from (10.1) are

$$J_\mu = \rho_e\,\frac{U_\mu}{c} \tag{10.2}$$

10.1 The Invariance of Charge Density

We intend now to show that charge density is invariant. First, we find the magnitude of the current density four-vector $\mathbf{J}$ from (10.1) in the unprimed frame to be

$$|J_\mu| = \frac{\rho_e}{c}\sqrt{\delta_{\mu\nu} U_\mu U_\nu} = \rho_e \tag{10.3}$$

We conclude, due to the invariance of the magnitude of all four-vectors, that the magnitude of the current density four-vector, or the charge density ρ_e, is the same in all inertial reference frames, or

$$\rho_e = \rho'_e \tag{10.4}$$

10.2 Transformation of Charge

From the invariance of charge density, we intend to show that charge is velocity dependent. The charge density in the primed frame is defined as

$$\rho_e' = \frac{q'}{v'} \tag{10.5}$$

where q' is the magnitude of the charge and v' is the volume containing the charge as measured by an observer at rest in the moving primed frame. Using the Jacobian J of $U_{\mu\nu}$, we can find the magnitude of the charge as measured by an observer at rest in the unprimed frame. Since we are making an instantaneous measurement of the volume, we take $\mu, \nu = 1, 2, 3$, so that

$$J = \frac{\partial(x, y, z)}{\partial(x', y', z')} = \frac{U_t}{c} = \gamma \tag{10.6}$$

Therefore, the volume in the unprimed frame is

$$v = J\,v' = \gamma v' \tag{10.7}$$

In this case, v' is the proper volume v_0. Using (10.4), (10.5), and (10.7), we have

$$\frac{q'}{v'} = \frac{q}{v} = \frac{q}{J\,v'} = \frac{q}{\gamma v'} \tag{10.8}$$

or

$$q = \gamma q' \tag{10.9}$$

This shows that the magnitude of a charge depends on its velocity. The magnitude of a charge at rest in a particular frame, in this case, q' is the *proper charge*. The proper (or rest) charge will be referred to, from this point on as q_0 and is the maximum magnitude of the charge. The *improper charge* q is defined as

$$q = \gamma q_0 \tag{10.10}$$

As can be seen from (10.10), if $U = c$, then $q = 0$.

11 The Potential Field

11.1 The Scalar Electric Potential

The *scalar electric potential* $\phi(x_1, y_1, z_1, t_1)$ at an arbitrary event $P_1(x_1, y_1, z_1, t_1)$, due to a *stationary* proper point charge $q_0(2)$ at the event $P_2(x_2, y_2, z_2, t_2)$ in the unprimed frame, using Gaussian units, is

$$\phi(1) = \frac{q_0(2)}{s_{12}} \tag{11.1}$$

where s_{12} is the spacetime interval between events $P(1)$ and $P(2)$ from (2.1).

To find the scalar potential $\phi(1)$ due to a *distribution* of stationary charge(s), we need to sum the contributions from each of the individual elements of charge. For the contribution of an element of proper charge $dq_0(2)$ at an event $P(2)$, we make use of the fact that $dq_0(2) =$

$\rho_e(2)dv_0(2)$, where $\rho_e(2)$ is the charge density at the event $P(2)$ and $dv_0(2) = dx_0dy_0dz_0$ is the element of proper volume containing $dq_0(2)$. The equation for the scalar potential $\phi(1)$ at an arbitrary event $P(1)$ due to a *stationary* distribution of charge(s) at the event $P(2)$ is

$$\phi(1) = \int \frac{\rho_e(2)dv_0(2)}{s_{12}} \tag{11.2}$$

11.2 The Potential Field Four-vector

The components of the *potential field four-vector* (or potential four-vector) $A_\mu = (A_1, A_2, A_3, A_4)$ (not to be confused with the arbitrary four-vector, $\mathbf{A}$) at the event $P(1)$, due to a distribution of *moving* charge(s) at the event $P(2)$, are

$$A_\mu(1) = \frac{1}{c}\int \frac{\rho_e(2)U_\mu dv_0(2)}{s_{12}} \tag{11.3}$$

where the U_μ are the components of the four-velocity of the element of charge $dq_0(2)$. Inserting (11.2) into (11.3) we find that, for a distribution of charge(s) moving at the same four-velocity U_μ,

$$A_\mu(1) = \phi(1)\,\frac{U_\mu}{c} \tag{11.4}$$

From (10.2) we can write

$$J_\mu(2) = \rho_e(2)\,\frac{U_\mu}{c} \tag{11.5}$$

Substituting (11.5) into (11.3), we can write the relation for the components of the potential field four-vector A_μ at an arbitrary event $P(1)$ due to a distribution of moving charge(s) at the event $P(2)$ as

$$A_\mu(1) = \int \frac{J_\mu(2)dv_0(2)}{s_{12}} \tag{11.6}$$

In the future, whenever we refer to the four-vector $\mathbf{A}$ (or A_μ), we will mean the potential field four-vector.

12 The Generalized Electric Field

Our intention, now, is to present the electric field in its most general form in terms of the derivatives of the potential field four-vector and to express the field, force, and energy-momentum equations in terms of the generalized electric field four-vector.

12.1 The Generalized Electric Field Four-vector

We express the *generalized electric field four-vector*, $\mathbf{E} = E_x\,\mathbf{e}_1 + E_y\,\mathbf{e}_2 + E_z\,\mathbf{e}_3 + E_t\,\mathbf{e}_4$, as the derivative product

$$\mathbf{E} = \mathbf{dA} \tag{12.1}$$

where $\mathbf{A}$ is the potential four-vector. We can write the components of $\mathbf{E}$ as

$$E_\mu = U_{\mu\nu}\partial_\nu\phi \tag{12.2}$$

where $U_{\mu\nu}$ is the transformation matrix (6.7) and ϕ is the static electric potential from (11.2). From this point on, we will refer to the generalized electric field four-vector as, simply, the electric field four-vector.

12.2 Correspondence with Conventional Fields

We can also write the electric field four-vector $\mathbf{E}$ as

$$\mathbf{E} = \mathbf{d} \cdot \mathbf{A} + \mathbf{d} \times \mathbf{A} + \mathbf{d} : \mathbf{A} \tag{12.3}$$

The spatial part of $\mathbf{E}$ from (12.3) is $\mathbf{d} \times \mathbf{A} + \mathbf{d} : \mathbf{A}$. Included in these terms are the conventional electric and magnetic field three-vectors $\mathbf{e}$ and $\mathbf{b}$, where $\mathbf{e} = \mathbf{d} : \mathbf{A}$ and $\mathbf{b} = \mathbf{d} \times \mathbf{A}$. It is important to remember that $\mathbf{d} : \mathbf{A}$, and $\mathbf{d} \times \mathbf{A}$ can be positive or negative independently of each other. Note that the electric field three-vector $\mathbf{e}$ should not be confused with the basis vectors $\mathbf{e}_\mu$ which will always be accompanied by a subscript.

In addition, $\mathbf{E}$ includes the time component of the electric field E_t, where $E_t \mathbf{e}_4 = \mathbf{d} \cdot \mathbf{A}$, which can also be independently positive or negative. We suspect that E_t contains the components of the gravitational field four-vector, $g_\mu = (g_x, g_y, g_z, g_t)$, where

$$g_x = \pm\partial_x A_x, \qquad g_y = \pm\partial_y A_y, \qquad g_z = \pm\partial_z A_z, \qquad g_t = \pm\partial_t A_t \tag{12.4}$$

We again suspect that the time component g_t of the gravitational field four-vector is the scalar nuclear field $n = g_t$.

12.3 The Electric Field Equations

We write the *electric field equations* as the derivative product

$$4\pi\mathbf{J} = \mathbf{dE} \tag{12.5}$$

where $\mathbf{J}$ is the current density four-vector from (10.1). Both the homogeneous and inhomogeneous Maxwell's electromagnetic field equations are included in (12.5). In addition, (12.5) includes terms containing the derivatives of the time component E_t of the electric field four-vector.

Inserting (12.1) into (12.5), we see that a possible representation of (12.5) reduces to

$$\mathbf{d}^2\mathbf{A} = -4\pi\mathbf{J} \tag{12.6}$$

or, in component form,

$$\partial^2 A_\mu = -4\pi J_\mu \tag{12.7}$$

13 The Derivative of the Current Density

The *derivative of the current density* $\mathbf{dJ}$ is a four-vector that we will call $\mathbf{G}$, where

$$\mathbf{G} = \mathbf{dJ} \tag{13.1}$$

We can express (13.1), using (12.5) as

$$4\pi\mathbf{G} = \mathbf{d}(\mathbf{dE}) \tag{13.2}$$

which can be written as

$$\mathbf{d}^2\mathbf{E} = -4\pi\mathbf{G} \tag{13.3}$$

where **E** is the electric field four-vector. In component form, we can write (13.3) as

$$\partial^2 E_\mu = -4\pi G_\mu \tag{13.4}$$

Note that a possible G_t component from (13.1) is

$$G_t = \partial_x J_x + \partial_y J_y + \partial_z J_z + \partial_t J_t = \partial_\mu J_\mu \tag{13.5}$$

which is the four-divergence of the current density. In conventional electromagnetic theory we have $\partial_\mu J_\mu = 0$. However, in this paper, it can be seen from (13.4) that

$$G_t = -\frac{1}{4\pi}\,\partial^2 E_t \tag{13.6}$$

which, in general, is nonzero.

14 The Energy-Momentum Density Four-vector

We define the *generalized energy-momentum density four-vector* **T** as

$$\mathbf{T} = \mathbf{JA} \tag{14.1}$$

where **J** is the current density four-vector and **A** is the potential four-vector.

Decomposing **J** and **A**, we get $\mathbf{J} = \rho_e \mathbf{U}/c$, where ρ_e is the charge density, and $\mathbf{A} = \phi\mathbf{U}/c$, where ϕ is the scalar electric potential, so that

$$\mathbf{JA} = \frac{\rho_e \phi}{c^2}\mathbf{UU} \tag{14.2}$$

We will refer to the quantity $\rho_e\phi/c^2$ as the *mass density* ρ_m, or

$$\rho_m = \frac{\rho_e \phi}{c^2} \tag{14.3}$$

since it has the units of mass density. The quantity

$$\mathbf{P} = \rho_m \mathbf{U} \tag{14.4}$$

will, then, be referred to as the *momentum density four-vector* **P**, so that (14.2) now becomes

$$\mathbf{JA} = \mathbf{PU} \tag{14.5}$$

Then substituting (14.5) into (14.1) we have an alternate form of (14.1)

$$\mathbf{T} = \mathbf{PU} \tag{14.6}$$

or in component form

$$T_\mu = c\,U_{\mu\nu} P_\nu \tag{14.7}$$

where the P_ν are the components of the momentum density four-vector.

15 Conservation of Energy-Momentum Density

Due to (4.20), we can write the derivative of $\mathbf{JA}$ as

$$\mathbf{d}((\mathbf{JA})) = (\mathbf{dJ})\mathbf{A} + \mathbf{J}(\mathbf{dA}) \tag{15.1}$$

After carrying out the multiplications on the right-hand side of (15.1), we can say that

$$(\mathbf{dJ})\mathbf{A} + \mathbf{J}(\mathbf{dA}) = 0 \tag{15.2}$$

Substituting (15.2) into (15.1), we get

$$\mathbf{d}((\mathbf{JA})) = 0 \tag{15.3}$$

After substituting (14.1) into (15.3), we get the equation for the *conservation of energy-momentum density* for an isolated system of particles

$$\mathbf{dT} = 0 \tag{15.4}$$

We can simplify (15.4) by introducing the four-vector $\mathbf{Z}$, which we define as

$$\mathbf{Z} = \mathbf{dT} \tag{15.5}$$

so that (15.4) can now be written as

$$\mathbf{Z} = 0 \tag{15.6}$$

A possible representation of (15.2) and (15.3), reduces to

$$\mathbf{d}(\mathbf{J} \cdot \mathbf{A}) = 0 \tag{15.7}$$

or,

$$\partial_\mu (J_\nu A_\nu) = 0 \tag{15.8}$$

16 The Force Density Four-vector

In the case of the electric field, the *force density four-vector* $\mathbf{F}$, or force per unit proper volume on a compact distribution of test charge(s) moving with current density $\mathbf{J}$ in an electric field $\mathbf{E}$ can be written as

$$\mathbf{F} = \mathbf{JE} \tag{16.1}$$

or, from (12.1), we can write (16.1) as

$$\mathbf{F} = \mathbf{J}(\mathbf{dA}) \tag{16.2}$$

Another way to express (16.1), using (12.5), is

$$\mathbf{F} = \frac{1}{4\pi}(\mathbf{dE})\mathbf{E} \tag{16.3}$$

Still another way to express (16.1) is in the component form

$$F_\mu = \rho_e U_{\mu\nu} E_\nu \tag{16.4}$$

17 The Equations of Motion

To get the equation of motion, we first write (15.2) as

$$-(\mathbf{dJ})\mathbf{A} = \mathbf{J}(\mathbf{dA}) \tag{17.1}$$

then, substituting (12.1), (13.1), and (16.1) into (17.1), we have

$$\mathbf{F} = -\mathbf{GA} = \mathbf{JE} \tag{17.2}$$

or, in component form,

$$F_\mu = -\phi\, U_{\mu\nu} G_\nu = \rho_e\, U_{\mu\nu} E_\nu \tag{17.3}$$

We recognize that the term on the right-hand side of (17.1) includes the conventional Lorentz force density. However, in addition to the usual terms, we get terms containing the time component E_t of the electric field that do not appear in the Lorentz equations.

Using (14.2) and (14.3), we can write $(\mathbf{dJ})\mathbf{A}$, on the left-hand side of (17.1), in the form

$$(\mathbf{dJ})\mathbf{A} = \rho_m (\mathbf{dU})\mathbf{U} \tag{17.4}$$

for $\rho_e =$ constant.

Inserting (12.1) and (17.4) into (17.1), we have

$$-(\mathbf{dU})\mathbf{U} = \frac{1}{\rho_m}\,\mathbf{JE} \tag{17.5}$$

Defining the *generalized derivative four-vector* $\mathbf{D}$ as

$$\mathbf{D} = -\mathbf{dU} \tag{17.6}$$

and inserting (17.6) into (17.5), we can write the generalized *equation of motion* as

$$\mathbf{DU} = \frac{1}{\rho_m}\,\mathbf{JE} \tag{17.7}$$

or, from (12.5),

$$\mathbf{DU} = \frac{1}{4\pi\rho_m}\,(\mathbf{dE})\mathbf{E} \tag{17.8}$$

We can also write (17.7), in component form, as

$$U_{\mu\nu} D_\nu = \frac{\rho_e}{c\,\rho_m}\, U_{\mu\nu} E_\nu \tag{17.9}$$

We notice that the time component of $\mathbf{D}$ can be written as

$$\begin{aligned} D_t &= \partial_x U_x + \partial_y U_y + \partial_z U_z + \partial_t U_t \\ &= \frac{\partial}{\partial x}\frac{dx}{d\tau} + \frac{\partial}{\partial y}\frac{dy}{d\tau} + \frac{\partial}{\partial z}\frac{dz}{d\tau} + \frac{\partial}{c\partial t}\frac{cdt}{d\tau} \\ &= \frac{d}{d\tau} \end{aligned} \tag{17.10}$$

which results in the components $D_t U_\mu$ in (17.7) taking the form of the conventional four-acceleration $a_\mu = dU_\mu/d\tau$.

18 The Energy-Momentum Tensor

We can define an *energy-momentum tensor* with components $T_{\mu\nu}$, as

$$T_{\mu\nu} = \frac{1}{4\pi}\left(E_\mu E_\nu - \frac{1}{2}\left(\epsilon_{\mu\nu\lambda\beta}E_\lambda E_\beta + \delta_{\mu\nu}E_\alpha E_\alpha\right)\right) \tag{18.1}$$

or

$$8\pi T_{\mu\nu} = 2\,E_\mu E_\nu - \epsilon_{\mu\nu\lambda\beta}E_\lambda E_\beta - \delta_{\mu\nu}E_\alpha E_\alpha \tag{18.2}$$

where $\epsilon_{\mu\nu\lambda\beta}$ is the four-dimensional permutation symbol and the E_μ are the components of the electric field four-vector. The set of components for the energy-momentum tensor $T_{\mu\nu}$ in (18.1) is only one of the possible sets of components for $T_{\mu\nu}$. By choosing alternate sets of rules for the products of basis vectors for $(\mathbf{dE})\mathbf{E}$ in (16.3), we will obtain different sets of components for $T_{\mu\nu}$.

It is important to remember that the E_μ in (18.1) and (18.2) are the components of the *generalized* electric field, thus they contain the components of the conventional electric, magnetic, and we suspect, the gravitational and nuclear fields as well. Note also, that the indices of the components of (18.1) and (18.2) range from 1 to 4, not 1 to 3, so that the energy-momentum tensor (18.1) has the same form, throughout, unlike the conventional electromagnetic energy-momentum tensor.

To show the conservation of energy, we set $\mu = 4$ in (18.1) and take the four-divergence, to get

$$\partial_\nu T_{4\nu} = \rho_e\, U_{4\nu} E_\nu \tag{18.3}$$

Similarly, we find the equations for the conservation of momentum by setting $\mu = 1, 2, 3$ in (18.1). Therefore, we can write the equation for the conservation of energy-momentum in terms of the energy-momentum tensor as

$$\partial_\nu T_{\mu\nu} = \rho_e\, U_{\mu\nu} E_\nu \tag{18.4}$$

for $\mu, \nu = 1, 2, 3, 4$.

Notice that the terms $\epsilon_{\mu\nu\lambda\beta}E_\lambda E_\beta$ in (18.1) vanish, but are "resurrected" when we take the four-divergence of $T_{\mu\nu}$, since

$$\partial_\nu(\epsilon_{\mu\nu\lambda\beta}E_\lambda E_\beta) = \epsilon_{\mu\nu\lambda\beta}\,\partial_\nu(E_\lambda E_\beta) = \epsilon_{\mu\nu\lambda\beta}(\partial_\nu E_\lambda)E_\beta + \epsilon_{\mu\lambda\nu\beta}E_\lambda(\partial_\nu E_\beta) \tag{18.5}$$

For example, if $\mu = 1$,

$$\begin{aligned}\partial_\nu(\epsilon_{1\nu\lambda\beta}E_\lambda E_\beta) = 2\,(&(\partial_2 E_3)E_4 - (\partial_2 E_4)E_3 + (\partial_3 E_4)E_2 \\ &- (\partial_3 E_2)E_4 + (\partial_4 E_2)E_3 - (\partial_4 E_3)E_2)\end{aligned}$$

which, in general, is nonzero.

In order to simplify (18.2), we introduce the tensor $E_{\mu\nu}$ with components

$$E_{\mu\nu} = 2\,E_\mu E_\nu - \epsilon_{\mu\nu\lambda\beta}E_\lambda E_\beta - \delta_{\mu\nu}E_\alpha E_\alpha \tag{18.6}$$

Inserting (18.6) into (18.2), we get

$$E_{\mu\nu} = 8\pi T_{\mu\nu} \tag{18.7}$$

19 The Work Density Four-vector

We introduce the *work density four-vector* $\mathbf{W}$, which is the work per unit proper volume done on a body by a constant force density $\mathbf{F}$ over a spacetime displacement $\mathbf{S}$ as the four-vector product

$$\mathbf{W} = \mathbf{F}\mathbf{S} \tag{19.1}$$

where $\mathbf{F}$ is the force density four-vector and $\mathbf{S}$ is the *displacement four-vector* from the event $P_1(x_1, y_1, z_1, t_1)$ to the event $P_2(x_2, y_2, z_2, t_2)$,

$$\mathbf{S} = S_x\,\mathbf{e}_1 + S_y\,\mathbf{e}_2 + S_z\,\mathbf{e}_3 + S_t\,\mathbf{e}_4 \tag{19.2}$$

where

$$S_x = x_2 - x_1, \qquad S_y = y_2 - y_1, \qquad S_z = z_2 - z_1, \qquad S_t = ct_2 - ct_1 \tag{19.3}$$

Included in the spatial part of $\mathbf{W}$ is the expression $\mathbf{F} \times \mathbf{S}$, which is related to the torque on a body, but also present is an expression for a new quantity $\mathbf{F} : \mathbf{S}$, which includes the impulse. The time part of $\mathbf{W}$ contains the expression $\mathbf{F} \cdot \mathbf{S}$, which includes the conventional, three-dimensional expression for work as well as the additional term $F_t S_t$.

We can write the *differential work* $d\mathbf{W}$ done by a variable force on a body, as

$$d\mathbf{W} = \mathbf{F}d\mathbf{s} \tag{19.4}$$

where $d\mathbf{s} = dx\,\mathbf{e}_1 + dy\,\mathbf{e}_2 + dz\,\mathbf{e}_3 + cdt\,\mathbf{e}_4$ is the *differential displacement four-vector* and $\mathbf{F}d\mathbf{s}$ is the four-vector product of $\mathbf{F}$ and $d\mathbf{s}$.[1] To find the work done by this variable force on the body, we integrate along its worldline from the event P_1 to the event P_2, to get

$$\mathbf{W} = \int_{P_1}^{P_2} \mathbf{F}d\mathbf{s} \tag{19.5}$$

In the case of the work done by a variable force on a current distribution in an electric field, we can write (19.4), using (16.2), as

$$d\mathbf{W} = (\mathbf{J}(\mathbf{dA}))d\mathbf{s} \tag{19.6}$$

We can then reduce (19.6) to

$$d\mathbf{W} = \mathbf{J}d\mathbf{A} \tag{19.7}$$

This implies that we can write the work density four-vector $\mathbf{W}$, in this case, as

$$\mathbf{W} = \mathbf{J}\mathbf{A} \tag{19.8}$$

The differential magnitude dW of $d\mathbf{W}$, from (19.7), is simply the magnitude of $\mathbf{J}d\mathbf{A}$, which is $\rho_e\,d\phi$, where ρ_e is the charge density on which the work is being done, and $d\phi$ is the differential electric potential at the location of ρ_e. Thus, we can write dW as,

$$dW = \rho_e\,d\phi \tag{19.9}$$

[1] The differential d is not to be confused with the derivative four-vector $\mathbf{d}$

We can now find the magnitude of the work per unit proper volume W done on ρ_e, by integrating (19.9) from the initial potential ϕ_1 to the final potential ϕ_2 at the location of ρ_e,

$$W = \int_{\phi_1}^{\phi_2} \rho_e \, d\phi \tag{19.10}$$

From (19.10) we see that the magnitude of the work per unit proper volume done on ρ_e is independent of the displacement of ρ_e. It depends only on the difference between the initial and final potentials at the location of ρ_e.

Evidently, work can be done on a body whether or not it undergoes a displacement in space. For example, in order to create a distribution of charged particles, work must be done on each particle to move it into place against the fields of the particles already in place (assembled particles). In addition, however, work must be done on the assembled particles, in order to keep them in place, against the field of each new particle, as the new particle is moved into place. This additional work done on the assembled particles in order to keep them in place must be included, along with the work initially done on each new particle to move it into place, in the total work required to create the distribution and thus in the total energy of the distribution.

20 The Angular Momentum Density Four-vector

The classical definition of angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. This is a three-dimensional representation which, as we would like to show, is part of a four-dimensional quantity.

The *angular momentum density four-vector* $\mathbf{L}$ is defined, here, as

$$\mathbf{L} = \mathbf{X}\mathbf{P} \tag{20.1}$$

where $\mathbf{X}$ is the position four-vector (6.1) and $\mathbf{P}$ is the momentum density four-vector (14.4). We can expand the right-hand side of (20.1) to get

$$\mathbf{X}\mathbf{P} = \mathbf{X} \cdot \mathbf{P} + \mathbf{X} \times \mathbf{P} + \mathbf{X} : \mathbf{P} \tag{20.2}$$

We see that the term $\mathbf{X} \times \mathbf{P}$ is the classical angular momentum, but we have in addition two terms $\mathbf{X} \cdot \mathbf{P}$ and $\mathbf{X} : \mathbf{P}$ which have no classical analogs in terms of angular momentum.

The term $\mathbf{X} \cdot \mathbf{P}$ resembles the phase of a wave function in momentum space. However, its significance in terms of angular momentum is not clear at this time, nevertheless, we believe that it represents some sort of, as yet unknown, oscillation or angular momentum density. The term $\mathbf{X} : \mathbf{P}$, we suspect, describes *spin angular momentum density.* This obviously contrasts with the quantum mechanical description of spin, however as can be seen, our description bears a closer resemblance to the classical representation of angular momentum, thus, we believe, it offers a more physical interpretation of spin than its quantum mechanical counterpart.

21 Covariant Formulation in General Coordinates

Until now, we have limited our study to four-dimensional rectangular coordinates in Euclidean spacetime. However, in order to describe our laws in general four-dimensional coordinate systems in Euclidean spacetime one must make several adjustments:

(i) The coordinates x_1, x_2, x_3, and x_4 must be written as x^1, x^2, x^3, and x^4 and now represent general coordinate systems.

(ii) In order to handle more general coordinate systems, every previous occurrence of the Kronecker delta $\delta_{\mu\nu}$ must be replaced by the metric tensor, $g_{\mu\nu}$.

(iii) Where previously it was unnecessary to distinguish between covariant and contravariant components of tensors, since there is no distinction between the two in rectangular coordinates, we must specify which we are using.

(iv) In rectangular coordinates, the derivative of a tensor is, simply, the ordinary derivative. But in general coordinates, we have terms which may include nonzero Christoffel symbols. These terms vanish in rectangular coordinates since the components of the metric tensor are constants. In general coordinates, however, the components of the metric tensor are not always constant, thus, the Christoffel symbols do not vanish, in general. Because of this, all ordinary derivatives of tensors must be replaced by their covariant derivatives.

22 Conclusions

We have described in our transformation equations a rotation of coordinates including an *automatic* rotation transverse to the direction of motion, which the Lorentz transformations do not describe. These rotations, however, are not necessarily observable as rotations, but as precession, time dilation, length contraction, etc.. In addition, as mentioned previously, an axis rotates in clockwise and counterclockwise directions, simultaneously. However, the clockwise rotation does not necessarily manifest physically in the same manner as the corresponding counterclockwise rotation.

In conventional theory, the magnitude of a charged particle is invariant. In this paper, the magnitude of a charge is reduced as its velocity is increased, however, the charge *density* in the region of the charge is invariant. Thus, the quantity of charge in a given volume does not change as long as no charge enters or leaves the region. It is the invariance of charge density, not the conservation of charge, which accounts for the neutrality of atoms. In addition, it is the variability of charge, rather than the increase of mass or momentum, that accounts for the reduced reaction of energetic charged particles to external fields in particle accelerators.

The definition of the scalar electric potential ϕ includes the spacetime interval s in the denominator, *not* the spatial interval r. This prevents an infinite potential at $r = 0$, a problem that plagues conventional electromagnetic theory.

It can be seen from the T_{44} component of (18.1) that the energy of the field can be negative. Rather than interpreting this as a liability, we suggest the possibility that the field of a particle *is* its antiparticle. We suspect that the creation of a particle travelling forward in time is accompanied by the creation of its antiparticle travelling backward in time.[2] Since an antiparticle travelling backward in time may be said to have negative energy and since the energy of the field from T_{44} is negative for $E_t^2 < E_x^2 + E_y^2 + E_z^2$, we suggest that the antiparticle travelling backward in time appears to us as the field of the particle. We are *not* referring, for example, to the particle/antiparticle pair created in the disintegration of an energetic photon. In that case, there is an electron and an anti-electron (positron) created. However, both of these "particles"

[2] We refer to particles (or antiparticles) travelling forward in time as particles, since they have the characteristics of particles, and particles (or antiparticles) travelling backward in time as antiparticles having the characteristics of fields.

are travelling *forward* in time. Associated with each of these particles, is a field which we claim is its antiparticle travelling *backward* in time. These particle/field (or particle/antiparticle) pairs are the pairs to which we refer. In this case, there are actually two particle/antiparticle pairs created. The electron and its field (antiparticle) comprise one particle/antiparticle pair, and the positron and its field (antiparticle) comprise the other particle/antiparticle pair. Since the antiparticles appear to us as (and are) the fields of the particles, one might even entertain the notion, that space and matter travelling backward in time are one and the same thing.

Due to this apparent particle/antiparticle link, we also conclude that there are *no* electric monopoles (since every charged particle is accompanied by its field), just as there are apparently no magnetic monopoles. These conclusions might offer a logical explanation for the puzzling absence of antimatter in the universe. If our suspicions are correct, this "missing" antimatter exists all around us as the fields of matter and, possibly, as space itself.

We have provided, here, what we consider to be a foundation upon which a deeper and more unified understanding of laws of nature might be built.