

# New Transformation Equations and the Electric Field Four-vector

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## Abstract

In special relativity, spacetime can be described as Minkowskian. We intend to show that spacetime, as well as the laws of electromagnetism, can be described using a four-dimensional Euclidean metric as a foundation. In order to formulate these laws successfully, however, it is necessary to extend the laws of electromagnetism by replacing the Maxwell tensor with an electric field four-vector. In addition, to assure the covariance of the new laws, we introduce equations that, completely, replace the Lorentz transformation equations and Lorentz group. The above replacements, we believe, lead naturally to a unification of the electromagnetic field with the gravitational and nuclear fields. We introduce, also, a new mathematical formalism which facilitates the presentation of our laws.

## 1 Introduction

Lorentz first derived his famous set of transformation equations from the electromagnetic field equations of Maxwell. They assure that Maxwell's equations will have the same form in any inertial frame of reference. Unfortunately, if Maxwell's equations are shown to be incomplete, then it is likely that the Lorentz equations are incorrect. We intend to show that this is the case.

Maxwell's equations are, essentially, a set of *three-dimensional* partial differential equations. That is, each equation contains the partial derivatives with respect to only *three* of the coordinates. In four-dimensional spacetime, a three-dimensional description of anything is inherently incomplete. We will extend Maxwell's equations so that they form a set of four-dimensional equations. In so doing, it is possible to encompass *all* of Maxwell's equations in a single vector equation by introducing an electric field four-vector. In addition to the electromagnetic field, we believe the new equation incorporates the gravitational and nuclear fields. This equation, however, is *not* Lorentz invariant and requires a new set of transformation equations in order that it has the same form in all inertial frames.

The Lorentz transformation equations forbid any contraction or expansion of coordinates transverse to the direction of motion. We present a new Euclidean set of transformation equations which *require* a rotation of the coordinates transverse to the direction of motion. There is also an analogous rotation in the plane described by the direction of motion and the time coordinate.

Due to the dependence of each of the Lorentz force equations on only three of the components of the four-velocity, they also form an incomplete set of equations. Therefore, we extend these equations to four-dimensions, as well. The equation of motion then follows naturally from our new force equation.

In expressing the force equations in terms of the fields, we arrive at an energy-momentum tensor with components which include the time component of our electric field four-vector. These components offer, among other things, a new description of the mechanism behind the flow of field energy.

A new mathematical formalism is introduced which substantially simplifies the expression of our laws and helps give a deeper understanding of the geometry behind them. This new formalism borrows its structure, in part, from Hamilton's quaternions and the Clifford algebras, but differs fundamentally from both.

## 2 Spacetime

In special relativity, spacetime can be described as Minkowskian. In this paper, we replace many of the laws of relativity and electromagnetism by using Euclidean spacetime as a foundation.

### 2.1 Events in Spacetime

We begin by introducing the concept of *events* in spacetime. These are the analogs in four-dimensional spacetime of points in three-dimensional space. An event is something that occurs at a specific place and at a specific time in a particular reference frame. We represent an event  $P$  in spacetime by  $P(x, y, z, t)$ , where  $x$ ,  $y$ ,  $z$ , and  $t$  are the coordinates of the event.

Events are measured by *observers* at rest in a particular reference frame who are present at the time and place of a specific event. Each observer has measured his distance from the origin of his frame by standard methods and carries a clock which has been synchronized with all other clocks in his frame by standard methods.

### 2.2 The Spacetime Interval

If we have two events  $P_1(x_1, y_1, z_1, t_1)$  and  $P_2(x_2, y_2, z_2, t_2)$ , or  $P(1)$  and  $P(2)$  for short, in a reference frame, the magnitude of the spacetime separation between the two events  $P(1)$  and  $P(2)$  is called the *spacetime interval*  $s_{12}$ , which is defined as

$$s_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + c^2(t_2 - t_1)^2} \quad (1)$$

where we have multiplied  $t_1$  and  $t_2$  by the invariant speed of light in vacuo  $c$ , to make the units consistent throughout. This is, simply, the extension of the Pythagorean theorem to four dimensions, with time being the fourth dimension. To simplify, we will sometimes drop the subscripts and refer to the spacetime interval as  $s$ . In terms of the square of the *element* of spacetime interval  $ds^2$ , (1) becomes

$$ds^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \quad (2)$$

An observer in a second reference frame might measure the two events  $P(1)$  and  $P(2)$  to be at  $P_1'(x_1', y_1', z_1', t_1')$  and  $P_2'(x_2', y_2', z_2', t_2')$  or  $P'(1)$  and  $P'(2)$ , respectively, in his frame. Notice,

that we have used primes (') above the coordinates, here, in order to distinguish the two sets of events in the two reference frames. In comparing quantities, we will frequently refer to the primed and unprimed quantities or frames of reference. The primed quantity will be indicated by a prime above the quantity (for example,  $x_1'$ ) and the unprimed quantity by the absence of a prime above the quantity (for example,  $x_1$ ). Observers in the primed frame would measure a spacetime interval

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 + c^2 dt'^2 \quad (3)$$

Both sets of observers, though they might measure the coordinates of the two events to be different, will agree on the spacetime interval between the events. Therefore, we can say that  $ds' = ds$  or

$$dx'^2 + dy'^2 + dz'^2 + c^2 dt'^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \quad (4)$$

To simplify, from now on, we will take  $P_1(0, 0, 0, 0)$  and  $P_1'(0, 0, 0, 0)$ , that is, the origins of the unprimed and primed frames coincide at  $t = t' = 0$ . We can, therefore, simply write  $P(2)$  and  $P'(2)$  as  $P(x, y, z, t)$  and  $P'(x', y', z', t')$  or  $P$  and  $P'$ , respectively.

### 2.3 The Spacetime Metric

We can write (2) in a more condensed form, by using the Einstein summation convention and the four-dimensional Euclidean *spacetime metric*  $g_{\mu\nu} = \delta_{\mu\nu}$ , where  $\delta_{\mu\nu}$  is the Kronecker delta,

$$\delta_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu \\ 0 & \text{for } \mu \neq \nu \end{cases} \quad (5)$$

Using (5), we can now write (2) as

$$ds^2 = \delta_{\mu\nu} dx_\mu dx_\nu \quad (\mu, \nu = 1, 2, 3, 4) \quad (6)$$

where

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ct \quad (7)$$

It is typical in rectangular coordinates to use subscripts throughout, rather than the usual subscripts *and* superscripts, since there is no distinction between the covariant and contravariant components of tensors. As in the Einstein summation convention, summation is to be carried out over the *repeated* indices in each term. The greek subscripts  $\mu, \nu, \dots$  will always range from 1 to 4, with 4 indicating time, unless otherwise noted. The latin subscripts  $i, j, k, \dots$  will range from 1 to 3 and will be used to indicate spatial components, only.

## 3 Proper Values

If the element of spacetime interval  $ds$  is between events that are separated solely by a time interval  $dt$  (the events occur at the same place) in the unprimed frame, that is,  $dx = dy = dz = 0$ , then from (2), we have  $ds = cdt$ . In this case  $dt$  is defined, in this paper, as the element of *proper time*  $d\tau$ , so that  $ds = cd\tau$ . If the situation is reversed, and  $dx' = dy' = dz' = 0$  in the primed frame, we can write (4) as

$$c^2 d\tau^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \quad (8)$$

Similarly, if the events are separated solely by a space interval (the events occur at the same time) in the unprimed frame, that is  $dt = 0$ , then from (2),  $ds = \sqrt{dx^2 + dy^2 + dz^2}$ . In this case,  $\sqrt{dx^2 + dy^2 + dz^2}$  is defined as the element of *proper length*  $d\lambda$ , therefore (2) becomes  $ds = d\lambda$ . If, on the other hand,  $dt' = 0$  in the primed frame, we can write (4) as

$$d\lambda^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \quad (9)$$

The proper time  $\tau$  and the proper length  $\lambda$  are always the *maximum* possible measurements of time and length made between events in any frame. We put no primes on the proper time or length, since they are the same in all reference frames. If there are space *and* time components of the spacetime interval between events in a reference frame, then *neither* component is proper and both will be less than the proper value. Therefore, we refer to them as *improper* values. However, an observer can always determine the proper values from his own measurements by using (8) or (9). In general, the proper value of *any* quantity, in this paper, will be its maximum value.

## 4 Four-vectors

We will represent an arbitrary *four-vector*  $\mathbf{A}$  by two equivalent methods. The first method, describing  $\mathbf{A}$  in terms of the basis vectors  $\mathbf{e}_\mu$  is

$$\mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3 + A_4 \mathbf{e}_4 \quad (10)$$

where

$$A_1 = A_x, \quad A_2 = A_y, \quad A_3 = A_z, \quad A_4 = A_t \quad (11)$$

or equivalently  $\mathbf{A} = A_\mu \mathbf{e}_\mu$ , where the  $A_\mu$  are the components of the four-vector  $\mathbf{A}$  in the directions of the basis vectors  $\mathbf{e}_\mu$ . The second method we will often use is

$$A_\mu = (A_1, A_2, A_3, A_4) \quad (12)$$

which is, simply, another way to express (10). Both of these methods will be used to represent four-vectors.

The *norm* or magnitude of our arbitrary four-vector  $|\mathbf{A}|$  is defined as

$$|\mathbf{A}| = \sqrt{\delta_{\mu\nu} A_\mu A_\nu} \quad (13)$$

### 4.1 The Multiplication of Four-vectors

We introduce now what we believe is a new mathematical formalism which we will use to simplify and condense our laws. These rules have been derived, in part, from Hamilton's quaternions, and the Clifford algebras.

#### 4.1.1 Basis Vectors

It is possible to multiply four-vectors algebraically by using the appropriate conventions for the products of the orthonormal *basis vectors*  $\mathbf{e}_\mu$ . We will represent all four-vectors and basis vectors,

in this paper, in bold face type. The basis vectors  $\mathbf{e}_\mu$  satisfy the relations

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for } i \neq j \quad (14)$$

$$\mathbf{e}_i\mathbf{e}_4 = \mp\mathbf{e}_4\mathbf{e}_i \quad (15)$$

$$\mathbf{e}_i\mathbf{e}_j = \pm\mathbf{e}_4\mathbf{e}_4 \quad \text{for } i = j \quad (16)$$

for  $i, j = 1, 2, 3$ . Note that the sign differences in (15) and (16) must be observed. That is, if we choose  $\mathbf{e}_i\mathbf{e}_4 = -\mathbf{e}_4\mathbf{e}_i$  in (15), we must choose  $\mathbf{e}_i\mathbf{e}_j = +\mathbf{e}_4\mathbf{e}_4$  in (16), and vice versa. However, whichever choice we make for (15) and (16), we can choose either  $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$  or  $-\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i$  for (14). The *orientation* (without regard for direction) of a product in (14) is along the line described by the unused basis vector. For example, the orientation of  $\mathbf{e}_1\mathbf{e}_2$  is along the line described by  $\mathbf{e}_3$ . For a product in (15), the orientation is along the line described by the  $\mathbf{e}_i$  basis vector. For example, the orientation of  $\mathbf{e}_1\mathbf{e}_4$  is along the line described by  $\mathbf{e}_1$ . The orientation of a product in (16) is along the worldline or timeline described by  $\mathbf{e}_4$ . For example, the orientation of  $\mathbf{e}_1\mathbf{e}_1$ , is along the worldline described by  $\mathbf{e}_4$ .

Although we have defined the orientations of the products, we have yet to define the *directions* of the products. One possible set of rules for the directions of the products of basis vectors, using the product rules, (14), (15), and (16), is

$$\begin{aligned} \mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_4 = -\mathbf{e}_4\mathbf{e}_1 = \mathbf{e}_1 & (17) \\ \mathbf{e}_3\mathbf{e}_1 &= -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_4 = -\mathbf{e}_4\mathbf{e}_2 = \mathbf{e}_2 \\ \mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3\mathbf{e}_4 = -\mathbf{e}_4\mathbf{e}_3 = \mathbf{e}_3 \\ \mathbf{e}_1\mathbf{e}_1 &= \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_3 = \mathbf{e}_4\mathbf{e}_4 = \mathbf{e}_4 \end{aligned}$$

where the result of each product is the basis vector on the right, for example, the result of the product  $\mathbf{e}_1\mathbf{e}_4$  in (17) is  $\mathbf{e}_1$ . The direction of this product is the direction of  $\mathbf{e}_1$ .

Another possible set of rules is

$$\begin{aligned} \mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_4 = \mathbf{e}_4\mathbf{e}_1 = \mathbf{e}_1 & (18) \\ \mathbf{e}_3\mathbf{e}_1 &= -\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_4 = \mathbf{e}_4\mathbf{e}_2 = \mathbf{e}_2 \\ \mathbf{e}_1\mathbf{e}_2 &= -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3\mathbf{e}_4 = \mathbf{e}_4\mathbf{e}_3 = \mathbf{e}_3 \\ -\mathbf{e}_1\mathbf{e}_1 &= -\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_3\mathbf{e}_3 = \mathbf{e}_4\mathbf{e}_4 = \mathbf{e}_4 \end{aligned}$$

Due to all of the possible permutations of (14), (15), and (16), there are *sixteen* possible sets of rules for a given four-vector product.

#### 4.1.2 The Vector Product

Using these rules, we can write the product of two four-vectors as a normal algebraic product. For example, the *vector product*  $\mathbf{AB}$  of two arbitrary four-vectors  $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 + A_4\mathbf{e}_4$  and  $\mathbf{B} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3 + B_4\mathbf{e}_4$  is

$$\mathbf{AB} = (A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 + A_4\mathbf{e}_4)(B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3 + B_4\mathbf{e}_4) \quad (19)$$

We will choose one of the eight sets of the rules for the products of basis vectors for our first example, but as will be shown later, multiple sets may be used in a given product. Let us choose,

for this example, the rules in (17). Multiplying (19) algebraically, using (17) for the products of the basis vectors, we get

$$\begin{aligned}
\mathbf{AB} &= (A_2B_3 - A_3B_2 + A_1B_4 - A_4B_1) \mathbf{e}_1 \\
&+ (A_3B_1 - A_1B_3 + A_2B_4 - A_4B_2) \mathbf{e}_2 \\
&+ (A_1B_2 - A_2B_1 + A_3B_4 - A_4B_3) \mathbf{e}_3 \\
&+ (A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4) \mathbf{e}_4
\end{aligned} \tag{20}$$

Consequently, the product of two four-vectors results in another four-vector. Note that we could have just as easily used (18), or any other of the sets of rules, rather than (17), for the product to obtain a different, but equally valid, result.

### 4.1.3 The Derivative Product

Using the same methods as above, we can write the *derivative product* of a four-vector (or derivative of a four-vector, for short) in vector form. For example, the derivative product of our arbitrary four-vector  $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 + A_4\mathbf{e}_4$  and the *derivative four-vector*,  $\mathbf{d} = \partial_1\mathbf{e}_1 + \partial_2\mathbf{e}_2 + \partial_3\mathbf{e}_3 + \partial_4\mathbf{e}_4$ , where

$$\partial_1 = \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_2}, \quad \partial_3 = \frac{\partial}{\partial x_3}, \quad \partial_4 = \frac{\partial}{\partial x_4} \tag{21}$$

can be written,

$$\mathbf{dA} = (\partial_1\mathbf{e}_1 + \partial_2\mathbf{e}_2 + \partial_3\mathbf{e}_3 + \partial_4\mathbf{e}_4)(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 + A_4\mathbf{e}_4) \tag{22}$$

The terms on the right-hand side of the derivative four-vector  $\mathbf{d}$  are *not* to be regarded as the partial derivatives of the basis vectors  $\mathbf{e}_\mu$ .

Multiplying, algebraically using the rules (17), as before, we get

$$\begin{aligned}
\mathbf{dA} &= (\partial_2A_3 - \partial_3A_2 + \partial_1A_4 - \partial_4A_1) \mathbf{e}_1 \\
&+ (\partial_3A_1 - \partial_1A_3 + \partial_2A_4 - \partial_4A_2) \mathbf{e}_2 \\
&+ (\partial_1A_2 - \partial_2A_1 + \partial_3A_4 - \partial_4A_3) \mathbf{e}_3 \\
&+ (\partial_1A_1 + \partial_2A_2 + \partial_3A_3 + \partial_4A_4) \mathbf{e}_4
\end{aligned} \tag{23}$$

The terms in parentheses in (23) *are* to be regarded as the partial derivatives of the  $A_\mu$ , that is

$$\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x_\mu} \tag{24}$$

since the  $A_\nu$  are the components of a four-vector.

### 4.1.4 Combined Products

As mentioned, previously, it is possible to combine sets of rules for the products of basis vectors in a single vector product. For example, we could combine (17) and (18), to get a combination

of the two sets of rules. The resulting signs of the terms are a *superposition* of the signs of *both* sets of rules. Using both sets of rules, (17) and (18), in (19), we have

$$\begin{aligned}\mathbf{AB} &= (A_2B_3 - A_3B_2 + A_1B_4 \mp A_4B_1) \mathbf{e}_1 \\ &+ (A_3B_1 - A_1B_3 + A_2B_4 \mp A_4B_2) \mathbf{e}_2 \\ &+ (A_1B_2 - A_2B_1 + A_3B_4 \mp A_4B_3) \mathbf{e}_3 \\ &+ (\pm A_1B_1 \pm A_2B_2 \pm A_3B_3 + A_4B_4) \mathbf{e}_4\end{aligned}\tag{25}$$

This is only one of the *combined products* possible. Other products can be created by combining multiple sets of rules in the manner of (25).

We will frequently use component notation along with vector notation in our descriptions. However, it is impossible to include all possible combinations of components contained in a given vector equation, in a single component equation. Therefore, any equation expressed in component form should, in general, be considered as only one possible form of the vector equation from which it was derived.

If we refer to the signs, “ $\pm$ ” and “ $\mp$ ”, in (25) as “plus over minus” and “minus over plus”, respectively, then the signs preceding the  $A_2B_3$ ,  $A_3B_1$ , and  $A_1B_2$  terms are, actually, “plus over plus”, and the signs preceding the  $A_3B_2$ ,  $A_1B_3$ , and  $A_2B_1$  terms are “minus over minus”. Unfortunately, there are no displayable mathematical symbols of this kind available.

In the case of terms preceded by “ $\pm$ ” or “ $\mp$ ”, the signs retain their opposite nature, even though each sign contains both “+” and “-” signs. For convenience, the terms preceded by “plus over plus” and “minus over minus” can be considered as “+” and “-”, respectively. The signs “ $\pm$ ”, “ $\mp$ ”, “plus over plus”, and “minus over minus”, will be referred to as *combined signs*.

#### 4.1.5 Vector Notation

The four-vector product  $\mathbf{AB}$  can be written more compactly, using *vector notation*, as

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \times \mathbf{B} + \mathbf{A} : \mathbf{B}\tag{26}$$

where

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (\pm A_1B_1 \pm A_2B_2 \pm A_3B_3 + A_4B_4) \mathbf{e}_4 \\ \mathbf{A} \times \mathbf{B} &= (A_2B_3 - A_3B_2) \mathbf{e}_1 + (A_3B_1 - A_1B_3) \mathbf{e}_2 + (A_1B_2 - A_2B_1) \mathbf{e}_3 \\ \mathbf{A} : \mathbf{B} &= (A_1B_4 \mp A_4B_1) \mathbf{e}_1 + (A_2B_4 \mp A_4B_2) \mathbf{e}_2 + (A_3B_4 \mp A_4B_3) \mathbf{e}_3\end{aligned}\tag{27}$$

The signs in the terms on the right-hand side of (27) reflect the product rules, (14), (15), and (16).

Just as in four-vector product, the derivative product  $\mathbf{dA}$  can be written in condensed form as

$$\mathbf{dA} = \mathbf{d} \cdot \mathbf{A} + \mathbf{d} \times \mathbf{A} + \mathbf{d} : \mathbf{A}\tag{28}$$

where

$$\begin{aligned}\mathbf{d} \cdot \mathbf{A} &= (\pm \partial_1 A_1 \pm \partial_2 A_2 \pm \partial_3 A_3 + \partial_4 A_4) \mathbf{e}_4 \\ \mathbf{d} \times \mathbf{A} &= (\partial_2 A_3 - \partial_3 A_2) \mathbf{e}_1 + (\partial_3 A_1 - \partial_1 A_3) \mathbf{e}_2 + (\partial_1 A_2 - \partial_2 A_1) \mathbf{e}_3 \\ \mathbf{d} : \mathbf{A} &= (\partial_1 A_4 \mp \partial_4 A_1) \mathbf{e}_1 + (\partial_2 A_4 \mp \partial_4 A_2) \mathbf{e}_2 + (\partial_3 A_4 \mp \partial_4 A_3) \mathbf{e}_3\end{aligned}\tag{29}$$

The product  $\mathbf{d} \cdot \mathbf{A}$  is the four-divergence of  $\mathbf{A}$ ,  $\mathbf{d} \times \mathbf{A}$  is the curl of  $\mathbf{A}$ , and  $\mathbf{d} : \mathbf{A}$  is a new product we will call the *evolution* of  $\mathbf{A}$ . Each product on the right-hand sides of (26) and (28) can be positive or negative independently of the other products, resulting in sixteen possible combinations for  $\mathbf{A}\mathbf{B}$  and  $\mathbf{d}\mathbf{A}$ .

## 5 The Four-velocity

Suppose that the primed frame of reference is in uniform motion with respect to the unprimed frame. Reference frames at rest or in uniform motion with respect to each other are referred to as *inertial* reference frames. This motion is represented in four-dimensional spacetime by the velocity four-vector, or *four-velocity*  $\mathbf{U}$ .

The components of the four-velocity  $U_\mu$  of the primed frame, according to an observer at rest in the unprimed frame (unprimed observer), are

$$U_\mu = \frac{dx_\mu}{d\tau} \quad (30)$$

where we have used the proper time  $\tau$  in the denominator rather than the coordinate time  $t$  in order that the components  $U_\mu$  form a four-vector. We represent the velocity four-vector by  $\mathbf{U} = U_\mu \mathbf{e}_\mu$  or, equivalently, by  $U_\mu = (U_1, U_2, U_3, U_4)$ . In the future we will assume, in general, that the unprimed frame of reference is at rest, and that the primed frame of reference is in uniform motion with respect to the unprimed frame. As we will show next, the magnitude of the four-velocity is invariant.

### 5.1 Invariant Magnitude of the Four-velocity

Let us imagine that two events occur at the same place, but at different times in the primed reference frame, that is,  $dx' = dy' = dz' = 0$ . Since an observer at rest in the primed frame (primed observer) measures no space interval between the events, his measurement of the space-time interval is entirely temporal. Therefore, as discussed in Section 3,

$$ds' = cd\tau \quad (31)$$

Therefore, from (8), we can write

$$c^2 d\tau^2 = dx^2 + dy^2 + dz^2 + c^2 dt^2 \quad (32)$$

Dividing both sides of (32) by  $d\tau^2$ , we get

$$c^2 = \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 + \left(\frac{cdt}{d\tau}\right)^2 \quad (33)$$

or, from (30) and (33),

$$c^2 = U_x^2 + U_y^2 + U_z^2 + U_t^2 \quad (34)$$

Now, the norm or magnitude of the four-velocity  $|\mathbf{U}|$ , using (13), is defined as

$$|\mathbf{U}| = \sqrt{\delta_{\mu\nu} U_\mu U_\nu} \quad (35)$$

but

$$\sqrt{\delta_{\mu\nu}U_\mu U_\nu} = \sqrt{U_x^2 + U_y^2 + U_z^2 + U_t^2} \quad (36)$$

so, from (34), (35), and (36),

$$|\mathbf{U}| = c \quad (37)$$

Since  $\mathbf{U}$  represents an arbitrary four-velocity, we conclude that the magnitude of the four-velocity of *any* body is the *invariant* speed of light,  $c$ . In the case of a body at *rest*, the four-velocity is  $U_\mu = (0, 0, 0, c)$ , where  $U_t = c$ .

## 6 The Transformation Equations

We wish to find a set of coordinate transformation equations that assure the covariance of the laws of physics described in this paper. Initially, we are making a transformation of coordinates from a stationary unprimed frame of reference to a uniformly moving primed frame, so we assume that the transformation involves the four-velocity  $U_\mu = (U_x, U_y, U_z, U_t)$  of the moving frame. But the transformed coordinates must have the same units as the original coordinates, therefore, we divide the  $U_\mu$  by the speed of light,  $c$ .

We take the origins of the unprimed and primed frames to coincide at  $t = t' = 0$  and the  $x$ ,  $y$ , and  $z$  axes to be parallel to the corresponding axes of the primed frame when both frames are at rest. Let us define the position four-vector  $\mathbf{X}$  in the unprimed frame, as

$$\mathbf{X} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + ct\mathbf{e}_4 \quad (38)$$

which is directed from the origin of the unprimed frame to an arbitrary event  $P(x, y, z, t)$  in the unprimed frame, and the position four-vector  $\mathbf{X}'$  in the primed frame, as

$$\mathbf{X}' = x'\mathbf{e}_1 + y'\mathbf{e}_2 + z'\mathbf{e}_3 + ct'\mathbf{e}_4 \quad (39)$$

directed from the origin of the primed frame to the same event  $P'(x', y', z', t')$  in the primed frame. A coordinate transformation is, essentially, the operation of transforming the four-vector  $\mathbf{X}$  into the four-vector  $\mathbf{X}'$ . This is accomplished through the four-vector product  $(1/c)\mathbf{U}\mathbf{X}$  or *transformation equation*,

$$\mathbf{X}' = \frac{1}{c}\mathbf{U}\mathbf{X} \quad (40)$$

Equation (40) contains all possible combinations of components resulting from all possible combined products of basis vectors, as discussed in Section 4.1. Unfortunately, it is impossible to show all possible combinations in component form, simultaneously. We can, however, expand the right-hand side of (40), using one of the possible combinations, to get

$$\begin{aligned} \mathbf{X}' &= (1/c)((\pm U_t x \pm U_z y \mp U_y z \mp U_x ct) \mathbf{e}_1 \\ &\quad + (\mp U_z x \pm U_t y \pm U_x z \mp U_y ct) \mathbf{e}_2 \\ &\quad + (\pm U_y x \mp U_x y \pm U_t z \mp U_z ct) \mathbf{e}_3 \\ &\quad + (\pm U_x x \pm U_y y \pm U_z z \pm U_t ct) \mathbf{e}_4) \end{aligned} \quad (41)$$

Equating components from the right-hand sides of (39) and (41), we find

$$\begin{aligned}
x' &= \frac{1}{c} (\pm U_t x \pm U_z y \mp U_y z \mp U_x ct) \\
y' &= \frac{1}{c} (\mp U_z x \pm U_t y \pm U_x z \mp U_y ct) \\
z' &= \frac{1}{c} (\pm U_y x \mp U_x y \pm U_t z \mp U_z ct) \\
ct' &= \frac{1}{c} (\pm U_x x \pm U_y y \pm U_z z \pm U_t ct)
\end{aligned} \tag{42}$$

or, in condensed form, (42) becomes

$$x'_\mu = U_{\mu\nu} x_\nu \tag{43}$$

where

$$U_{\mu\nu} = \frac{1}{c} \begin{pmatrix} \pm U_t & \pm U_z & \mp U_y & \mp U_x \\ \mp U_z & \pm U_t & \pm U_x & \mp U_y \\ \pm U_y & \mp U_x & \pm U_t & \mp U_z \\ \pm U_x & \pm U_y & \pm U_z & \pm U_t \end{pmatrix} \tag{44}$$

The matrix  $U_{\mu\nu}$  in (44) (not to be confused with the velocity four-vector  $\mathbf{U}$  with components  $U_\mu$ ) will be referred to as a *transformation matrix*. Of course, there are other possible choices for the components of  $U_{\mu\nu}$  resulting from the use of alternate combined products of basis vectors. For example, without altering the signs of the other terms, we could have reversed the signs of  $U_{12}$ ,  $U_{13}$ ,  $U_{21}$ ,  $U_{23}$ ,  $U_{31}$ , and  $U_{32}$  in (44), while preserving the orthogonality of  $U_{\mu\nu}$ , as required in Euclidean spacetime. The equation (40) should be seen to contain *all* possible combinations which leave the Euclidean spacetime interval (1) invariant.

We can get a feeling for the geometrical meaning of (40) by giving a simplified example. Imagine that the primed frame is in uniform motion with four-velocity  $U_\mu = (U_x, 0, 0, U_t)$  relative to the unprimed frame. To simplify, we will consider only the positive  $U_t$  components in (44), here, although the negative  $U_t$  components are equally significant. In this case, (42) becomes

$$\begin{aligned}
x' &= \frac{1}{c} (U_t x \mp U_x ct) \\
y' &= \frac{1}{c} (U_t y \pm U_x z) \\
z' &= \frac{1}{c} (\mp U_x y + U_t z) \\
ct' &= \frac{1}{c} (\pm U_x x + U_t ct)
\end{aligned} \tag{45}$$

We see that the  $y'$ - $z'$  plane is rotated in clockwise *and* counterclockwise directions in the  $y$ - $z$  plane, *simultaneously*, by the angle  $\theta = \arctan(U_x/U_t)$  and that the  $x'$ - $t'$  plane is also similarly rotated in the  $x$ - $t$  plane. These results are for two frames which are both moving in the same direction “in time”, that is, events which are not simultaneous in either frame occur in the same order in both frames. However, in order for this to be the case, the coordinates of an event that occurs at a well defined time and place in one frame, must necessarily *not* be well defined in a second frame which is moving uniformly with respect to the first frame.

If we simplify (45) still further by setting  $y = z = 0$ , we have

$$\begin{aligned}x' &= \frac{1}{c} (U_t x \mp U_x ct) \\ ct' &= \frac{1}{c} (\pm U_x x + U_t ct)\end{aligned}\tag{46}$$

We see that the coordinate  $x'$  in (46) is not well defined due to the term  $\mp U_x ct$ . Similarly, the coordinate  $ct'$  is not well defined due to the term  $\pm U_x x$ . This results in a “spreading out” of the event  $P'$ . Thus, in the primed frame, there is no *specific* time and place that can be assigned to the event  $P'$  (unless  $U_x = 0$ ). We can say only that it began at the coordinates  $t'_1 = (1/c^2)(U_t ct - U_x x)$  and  $x'_1 = (1/c)(U_t x + U_x ct)$ , and that it ended at the coordinates  $t'_2 = (1/c^2)(U_t ct + U_x x)$  and  $x'_2 = (1/c)(U_t x - U_x ct)$ .

The spatial interval  $\Delta x'$  over which  $P'$  occurs is

$$\Delta x' = x'_2 - x'_1 = \frac{1}{c} ((U_t x - U_x ct) - (U_t x + U_x ct)) = -2U_x t\tag{47}$$

The temporal interval  $\Delta t'$  over which  $P'$  occurs is

$$\Delta t' = t'_2 - t'_1 = \frac{1}{c^2} ((U_t ct + U_x x) + (U_t ct - U_x x)) = \frac{2}{c} U_t t\tag{48}$$

Note that we have added the components  $t'_1$  and  $t'_2$ , in (48), in opposition to the usual method of subtraction used to find an interval, as in (47). We must also remember that we are not measuring the interval *between* events, in the case of  $P'$ , but the interval over which a *single* event occurs.

We should be able to derive the four-velocity  $U'_x$  of the unprimed frame relative to the primed frame from  $\Delta x'$  and  $\Delta t'$ . Using (47) and (48), we get

$$U'_x = \frac{\Delta x'}{\Delta \tau} = \frac{\Delta x'}{\Delta t'} \frac{\Delta t'}{\Delta \tau} = \frac{-2U_x t}{(2/c) U_t t} \frac{U_t}{c} = -c \frac{U_x}{U_t} \frac{U_t}{c} = -U_x\tag{49}$$

where the equality  $\Delta t'/\Delta \tau = U_t/c$  will be proven in Section 8. We see from (49), that the four-velocity  $U'_x$  of the unprimed frame relative to the primed frame is the negative of the four-velocity  $U_x$  of the primed frame relative to the unprimed frame, as required. The only exception is for  $t = 0$ , that is, when the two events are simultaneous in the unprimed frame.

## 6.1 Inverse Transformation Equations

We can find the inverse transformation equations, that is, the equations describing the transformation of coordinates from the primed frame to the unprimed frame, by remembering that, according to a primed observer, the unprimed frame is moving in the opposite direction. Therefore, by substituting  $U'_\mu = (-U_x, -U_y, -U_z, U_t)$  for the four-velocity of the unprimed frame relative to the primed frame and switching the four-vectors  $\mathbf{X}$  and  $\mathbf{X}'$  in (40) we get the *inverse transformation equation*

$$\mathbf{X} = \frac{1}{c} \mathbf{U}' \mathbf{X}'\tag{50}$$

or in component form,

$$x_\nu = x'_\mu U'_{\mu\nu}\tag{51}$$

*The transformation equations (40), (43), (50), and (51) REPLACE the Lorentz transformation equations and the Lorentz group.*

## 7 Transformation of Length

Assume that the primed frame is in uniform motion with four-velocity  $U_\mu = (U_x, 0, 0, U_t)$  relative to the unprimed frame. In order to compare measurements of the spatial interval between events in the direction of motion in the two frames, we take the interval between the origin and the event  $P'(x', 0, 0, 0)$  in the primed frame, where  $x'$  in this case is the proper length  $\lambda$  since  $y' = z' = ct' = 0$ . We wish to find the coordinates of the same event  $P(x, y, z, t)$ , in the unprimed frame. To transform coordinates between the primed frame and the unprimed frames, we will use (51). Expanding (51), using  $U_\mu$  and  $P'$  above, we have

$$\begin{aligned} x &= \frac{1}{c} (U_t x') \\ ct &= \frac{1}{c} (\pm U_x x') \end{aligned} \quad (52)$$

Since we are comparing spatial intervals, we are interested in the  $x$ -coordinate in the unprimed frame, or

$$x = \frac{U_t}{c} x' \quad (53)$$

The  $U_t/c$  part of (53) can be put in a more familiar form by remembering from (34) that

$$c^2 = U_x^2 + U_y^2 + U_z^2 + U_t^2 \quad (54)$$

After rearranging terms, we have

$$\frac{U_t}{c} = \sqrt{1 - U^2/c^2} \quad (55)$$

where

$$U^2 = U_x^2 + U_y^2 + U_z^2 \quad (56)$$

and inserting (55) into (53) we get

$$x = x' \sqrt{1 - U^2/c^2} \quad (57)$$

The coordinate  $x$  in (57) is the improper length measured by the unprimed observers and is less than the primed observers' measurement of the proper length  $x'$ . We see from (52) that the  $x$  coordinate of the event  $P$  is well defined, that is, it occurs at a specific place in the unprimed frame. However, the  $t$  coordinate of  $P$  in (52) is not clearly defined, it is "spread out" over the extended time interval  $-U_x x'/c^2 \leq t \leq U_x x'/c^2$ . That is, it does not occur at a specific instant in the unprimed frame.

The effect is reciprocal since, using (43) and  $P(x, 0, 0, 0)$  with  $x$  as our proper length, we obtain

$$x' = x \sqrt{1 - U^2/c^2} \quad (58)$$

## 8 Transformation of Time

We can use similar methods to compare elapsed times between events in two reference frames. Using the four-velocity  $U_\mu = (U_x, 0, 0, U_t)$  from Section 7 and an event  $P'(0, 0, 0, t')$  in the primed

frame, we employ (51) again, to find the coordinates of the event  $P(x, y, z, t)$ , in the unprimed frame, of the same event  $P'$  in the primed frame, to get

$$\begin{aligned} x &= \frac{1}{c} (\mp U_x ct') \\ ct &= \frac{1}{c} (U_t ct') \end{aligned} \quad (59)$$

But since we are comparing time measurements, we consider

$$t = \frac{U_t}{c} t' \quad (60)$$

or, inserting (55) into (60), we have

$$t = t' \sqrt{1 - U^2/c^2} \quad (61)$$

The coordinate  $t'$  in (61), in this case, is the proper time  $\tau$ , since  $x' = y' = z' = 0$  in the primed frame and the coordinate  $t$  in (61) is the improper time.

We can see from (59) that the  $t$  coordinate of the event  $P$  is well defined, that is, it occurs at a specific time in the unprimed frame. However, in this case, it is the  $x$  coordinate that is not clearly defined. The  $x$  coordinate of  $P$  in the unprimed frame, is “spread out” over the extended spatial interval  $-U_x t' \leq x \leq U_x t'$ . That is, it does not occur at a specific place in the unprimed frame.

Again, the effect is reciprocal.

## 9 The Transformation of Velocity

If we have a body in uniform motion relative to the primed frame, and the primed frame is in uniform motion relative to the unprimed frame, an observer in the primed frame can find the bodies motion relative the unprimed frame by using the equations for the inverse transformation of coordinates (50) or (51) with the appropriate substitutions. From this point on, in order to simplify, we will display only one of the signs from the combined sign of each term.

Imagine that the body is moving with uniform four-velocity  $V'_\mu = (V'_x, V'_y, V'_z, V'_t)$  relative to the primed frame and that the unknown four-velocity of the body relative to the unprimed frame is  $V_\mu = (V_x, V_y, V_z, V_t)$ . The primed frame, in turn, is moving with uniform four-velocity  $U_\mu = (U_x, U_y, U_z, U_t)$  relative to the unprimed frame. The primed observer can find  $\mathbf{V}$  by substituting  $\mathbf{V}$  and  $\mathbf{V}'$  for  $\mathbf{X}$  and  $\mathbf{X}'$ , respectively, in (50) to get

$$\mathbf{V} = \frac{1}{c} \mathbf{U} \mathbf{V}' \quad (62)$$

or alternately, we can make the same substitutions  $V_\nu$  and  $V'_\mu$  in place of  $x_\nu$  and  $x'_\mu$ , respectively, in (51) to obtain the same results in component form

$$V_\nu = V'_\mu U_{\mu\nu} \quad (63)$$

For  $\mathbf{U}$  and  $\mathbf{V}'$  in the same direction, for example  $U_\mu = (U_x, 0, 0, U_t)$  and  $V'_\mu = (V'_x, 0, 0, V'_t)$  and using (63), we get

$$\begin{aligned} V_x &= \frac{1}{c} (U_t V'_x + U_x V'_t) \\ V_t &= \frac{1}{c} (-U_x V'_x + U_t V'_t) \end{aligned} \quad (64)$$

For  $U_x \ll c$  and  $V'_x \ll c$ , we have  $U_t \approx c$  and  $V'_t \approx c$ , so that we get, approximately, the Galilean result  $V_x \approx V'_x + U_x$ .

In the case of  $U_\mu = (c, 0, 0, 0)$  and  $V'_\mu = (c, 0, 0, 0)$  we get  $V_x = V_y = V_z = 0$  and  $V_t = -c$  or  $V_\mu = (0, 0, 0, -c)$ .

## 10 The Current Density Four-vector

The *current density four-vector*  $\mathbf{J}$  for a distribution of charges moving with four-velocity  $\mathbf{U}$  is defined as

$$\mathbf{J} = \frac{\rho}{c} \mathbf{U} \quad (65)$$

where  $\rho$  is the *charge density* of the distribution. The components  $J_\mu$  of the current density four-vector from (65) are

$$J_\mu = \rho \frac{U_\mu}{c} \quad (66)$$

### 10.1 The Invariance of Charge Density

We intend now to show that charge density is invariant. First, we find the magnitude of the current density four-vector  $\mathbf{J}$  from (65) in the unprimed frame to be

$$|J_\mu| = \frac{\rho}{c} \sqrt{\delta_{\mu\nu} U_\mu U_\nu} = \rho \quad (67)$$

We conclude, due to the invariance of the magnitude of all four-vectors, that the magnitude of the current density four-vector, or the charge density  $\rho$ , is the same in all inertial reference frames, or

$$\rho = \rho' \quad (68)$$

### 10.2 Transformation of Charge

From the invariance of charge density, we intend to show that charge is velocity dependent. The charge density in the primed frame is defined as

$$\rho' = \frac{q'}{v'} \quad (69)$$

where  $q'$  is the magnitude of the charge and  $v'$  is the volume containing the charge as measured by an observer at rest in the moving primed frame. Using the Jacobian  $J$  of  $U_{\mu\nu}$ , we can find

the magnitude of the charge as measured by an observer at rest in the unprimed frame. Since we are making an instantaneous measurement of the volume, we take  $\mu, \nu = 1, 2, 3$ , so that

$$J = \frac{\partial(x, y, z)}{\partial(x', y', z')} = \frac{U_t}{c} = \sqrt{1 - U^2/c^2} \quad (70)$$

Therefore, the volume in the unprimed frame is

$$v = J v' = v' \sqrt{1 - U^2/c^2} \quad (71)$$

In this case,  $v'$  is the proper volume  $v_0$ . Using (68), (69), and (71), we have

$$\frac{q'}{v'} = \frac{q}{v} = \frac{q}{J v'} = \frac{q}{v' \sqrt{1 - U^2/c^2}} \quad (72)$$

or

$$q = q' \sqrt{1 - U^2/c^2} \quad (73)$$

This shows that the magnitude of a charge depends on its velocity. The magnitude of a charge at rest in a particular frame, in this case,  $q'$  is the *proper charge*. The proper (or rest) charge will be referred to, from this point on as  $q_0$  and is the maximum magnitude of the charge. The *improper charge*  $q$  is defined as

$$q = q_0 \sqrt{1 - U^2/c^2} \quad (74)$$

As can be seen from (74), if  $U = c$ , then  $q = 0$ .

## 11 The Potential Field

### 11.1 The Scalar Electric Potential

The *scalar electric potential*  $\phi(x_1, y_1, z_1, t_1)$  at an arbitrary event  $P_1(x_1, y_1, z_1, t_1)$ , due to a *stationary* proper point charge  $q_0(2)$  at the event  $P_2(x_2, y_2, z_2, t_2)$  in the unprimed frame is

$$\phi(1) = \frac{q_0(2)}{4\pi\epsilon_0 s_{12}} \quad (75)$$

where  $s_{12}$  is the spacetime interval between events  $P(1)$  and  $P(2)$  from (1) and  $\epsilon_0$  is the permittivity constant.

To find the scalar potential  $\phi(1)$  due to a *distribution* of stationary charges, we need to sum the contributions from each of the individual elements of charge. For the contribution of an element of proper charge  $dq_0(2)$  at an event  $P(2)$ , we make use of the fact that  $dq_0(2) = \rho(2)dv_0(2)$ , where  $\rho(2)$  is the charge density at the event  $P(2)$  and  $dv_0(2) = dx_0 dy_0 dz_0$  is the element of proper volume containing  $dq_0(2)$ . The equation for the scalar potential  $\phi(1)$  at an arbitrary event  $P(1)$  due to a distribution of *stationary* charges at the event  $P(2)$  is

$$\phi(1) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(2)dv_0(2)}{s_{12}} \quad (76)$$

## 11.2 The Potential Field Four-vector

The components of the *potential field four-vector* (or potential four-vector)  $A_\mu = (A_1, A_2, A_3, A_4)$  (not to be confused with the arbitrary four-vector,  $\mathbf{A}$ ) at the event  $P(1)$ , due to a distribution of *moving* charges at the event  $P(2)$ , are

$$A_\mu(1) = \frac{1}{4\pi\epsilon_0 c} \int \frac{\rho(2)U_\mu dv_0(2)}{s_{12}} \quad (77)$$

where the  $U_\mu$  are the components of the four-velocity of the element of charge  $dq_0(2)$ . Inserting (76) into (77) we find that, for a distribution of charges moving with uniform four-velocity  $U_\mu$ ,

$$A_\mu(1) = \phi(1) \frac{U_\mu}{c} \quad (78)$$

From (66) we can write

$$J_\mu(2) = \rho(2) \frac{U_\mu}{c} \quad (79)$$

Substituting (79) into (77), we can write the relation for the components of the potential field four-vector  $A_\mu$  at an arbitrary event  $P(1)$  due to a distribution of moving charges at the event  $P(2)$  as

$$A_\mu(1) = \frac{1}{4\pi\epsilon_0} \int \frac{J_\mu(2)dv_0(2)}{s_{12}} \quad (80)$$

In the future, whenever we refer to the four-vector  $\mathbf{A}$  (or  $A_\mu$ ), we will mean the potential field four-vector.

## 12 The Generalized Electric Field

Our intention, now, is to present the electric field in its most general form and to express the field, force, and energy-momentum equations in terms of the generalized electric field four-vector.

### 12.1 The Generalized Electric Field Four-vector

We express the *generalized electric field four-vector*,  $\mathbf{E} = E_x\mathbf{e}_1 + E_y\mathbf{e}_2 + E_z\mathbf{e}_3 + E_t\mathbf{e}_4$ , as the derivative product

$$\mathbf{E} = d\mathbf{A} \quad (81)$$

where  $\mathbf{A}$  is the potential four-vector. We can write the components of  $\mathbf{E}$  as

$$E_\mu = U_{\mu\nu}\phi_{,\nu} \quad (82)$$

where  $U_{\mu\nu}$  is the transformation matrix (44),  $\phi$  is the static electric potential from (76), and “ $\phi_{,\nu}$ ” indicates the partial derivative of  $\phi$  with respect to  $x_\nu$ . From this point on, we will refer to the generalized electric field four-vector as, simply, the electric field four-vector.

## 12.2 Correspondence with Conventional Fields

We can also write the electric field four-vector  $\mathbf{E}$  as

$$\mathbf{E} = \mathbf{d} \cdot \mathbf{A} + \mathbf{d} \times \mathbf{A} + \mathbf{d} : \mathbf{A} \quad (83)$$

The spatial part of  $\mathbf{E}$  from (83) is  $\mathbf{d} \times \mathbf{A} + \mathbf{d} : \mathbf{A}$ . As can be seen, this includes the conventional electric and magnetic field three-vectors  $\mathbf{e}$  and  $\mathbf{b}$ , respectively, where  $\mathbf{e} = \mathbf{d} : \mathbf{A}$  and  $\mathbf{b} = \mathbf{d} \times \mathbf{A}$ . It is important to remember that  $\mathbf{d} : \mathbf{A}$ , and  $\mathbf{d} \times \mathbf{A}$  can be positive or negative independently of each other. Note that the electric field three-vector  $\mathbf{e}$  should not be confused with the basis vectors  $\mathbf{e}_\mu$  which will always be accompanied by a subscript.

In addition,  $\mathbf{E}$  includes the time component of the electric field  $E_t$ , where  $E_t \mathbf{e}_4 = \mathbf{d} \cdot \mathbf{A}$ , which can also be independently positive or negative. We suspect that  $E_t$  contains the components of the gravitational field four-vector,  $g_\mu = (g_x, g_y, g_z, g_t)$ , where

$$\begin{aligned} g_x &= -\partial_x A_x \\ g_y &= -\partial_y A_y \\ g_z &= -\partial_z A_z \\ g_t &= -\partial_t A_t \end{aligned} \quad (84)$$

We again suspect that the time component  $g_t$  of the gravitational field four-vector is the scalar nuclear field  $n = -g_t = \partial_t A_t$ .

## 12.3 The Electric Field Equations

We write the *electric field equations* as the derivative product

$$4\pi \mathbf{J} = \mathbf{dE} \quad (85)$$

where  $\mathbf{J}$  is the current density four-vector from (65). We can also express (85) using (81) as

$$4\pi \mathbf{J} = \mathbf{d}(\mathbf{dA}) \quad (86)$$

where  $\mathbf{A}$  is the potential field four-vector. Both the homogeneous and inhomogeneous Maxwell's electromagnetic field equations are included in (85) as well as terms containing the derivatives of the time component  $E_t$  of the electric field four-vector.

In terms of the four-potential  $\mathbf{A}$ , we get as one possible result from (86)

$$-4\pi J_\mu = \partial^2 A_\mu \quad (87)$$

where

$$\partial^2 = \partial_\nu \partial_\nu = \frac{\partial^2}{\partial x_\nu^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{c^2 \partial t^2} \quad (88)$$

## 12.4 The Derivative of the Current Density

The *derivative of the current density*  $\mathbf{dJ}$  is an unknown four-vector which we will call  $\mathbf{G}$ , where

$$\mathbf{G} = \mathbf{dJ} \quad (89)$$

or, inserting (85) into (89), we have

$$4\pi\mathbf{G} = \mathbf{d}(\mathbf{dE}) \quad (90)$$

One possible representation of (90), in component form, is

$$4\pi G_\mu = \partial^2 E_\mu \quad (91)$$

Note that a possible  $G_t$  component from (89) is

$$G_t = \partial_x J_x + \partial_y J_y + \partial_z J_z + \partial_t J_t = \partial_\mu J_\mu \quad (92)$$

which is the four-divergence of the current density. In conventional electromagnetic theory  $\partial_\mu J_\mu = 0$ . However, in this paper, as can be seen from (91), we have

$$G_t = \frac{1}{4\pi} \partial^2 E_t \quad (93)$$

which, in general, is nonzero.

## 13 The Force Density Four-vector

For a compact distribution of test charges moving with current density  $\mathbf{J}$  in an electric field  $\mathbf{E}$ , we can write the *force density four-vector*  $\mathbf{F}$ , or force per unit proper volume on the distribution, as

$$\mathbf{F} = \mathbf{J}\mathbf{E} \quad (94)$$

Another way to express  $\mathbf{F}$  is in the component form

$$F_\mu = \rho U_{\mu\nu} E_\nu \quad (95)$$

A possible representation of (94), reduces to

$$\mathbf{F} = \mathbf{d}(\mathbf{J} \cdot \mathbf{A}) \quad (96)$$

or in component form

$$F_\mu = (J_\nu A_\nu)_{,\mu} \quad (97)$$

The conventional Lorentz force density equations are included in (94) and (95). However, in addition to the usual terms, we get terms containing the time component  $E_t$  of the electric field that do not appear in the Lorentz equations. We also get a set of equations which are the counterparts of the homogeneous Maxwell equations, i.e.  $\mathbf{j} \cdot \mathbf{b} = 0$  and  $\mathbf{j} \times \mathbf{e} + J_t \mathbf{b} = 0$ <sup>1</sup>, where  $\mathbf{j}$  is the current density three-vector  $\mathbf{j} = J_x \mathbf{e}_1 + J_y \mathbf{e}_2 + J_z \mathbf{e}_3$ .

<sup>1</sup>It is interesting to note that these two equations are independent of the charge density associated with  $\mathbf{J}$  and, in fact,  $\mathbf{J}$  can be replaced, entirely, by the four-velocity,  $\mathbf{U}$ , to obtain the equations  $\mathbf{u} \cdot \mathbf{b} = 0$  and  $\mathbf{u} \times \mathbf{e} + U_t \mathbf{b} = 0$ , where  $\mathbf{u} = U_x \mathbf{e}_1 + U_y \mathbf{e}_2 + U_z \mathbf{e}_3$ .

### 13.1 The Equations of Motion

The *equation of motion* for a compact distribution of charged test particles with current density  $\mathbf{J}$  moving in an electric field  $\mathbf{E}$  is

$$\mathbf{F} = \frac{d\mathbf{p}}{d\tau} = \mathbf{J}\mathbf{E} \quad (98)$$

where  $\mathbf{p}$  is the momentum density four-vector,

$$\mathbf{p} = \rho_m \mathbf{U} \quad (99)$$

and  $\rho_m$  is the mass density or proper mass  $m_0$  per unit proper volume  $\rho_m = m_0/v_0$  of the distribution of charges.

Using (96), we can write one possible representation of (98) as

$$\mathbf{F} = \frac{d\mathbf{p}}{d\tau} = \mathbf{d}(\mathbf{J} \cdot \mathbf{A}) \quad (100)$$

or, in component form using (97),

$$F_\mu = \frac{dp_\mu}{d\tau} = (J_\nu A_\nu)_{,\mu} \quad (101)$$

where the  $p_\mu$  are the components of the momentum density four-vector  $p_\mu = \rho_m U_\mu$ .

We can put (98) entirely in terms of the electric field four-vector  $\mathbf{E}$  by substituting (85) in place of  $\mathbf{J}$  to get

$$\mathbf{F} = \frac{d\mathbf{p}}{d\tau} = \frac{1}{4\pi} (\mathbf{dE})\mathbf{E} \quad (102)$$

### 13.2 The Energy-Momentum Tensor

We can define an *energy-momentum tensor*  $\mathbf{T}$  with components  $T_{\mu\nu}$ , as

$$T_{\mu\nu} = \frac{1}{4\pi} (E_\mu E_\nu - \frac{1}{2} (\varepsilon_{\mu\nu\lambda\beta} E_\lambda E_\beta + \delta_{\mu\nu} E_\lambda E_\lambda)) \quad (103)$$

or

$$8\pi T_{\mu\nu} = 2E_\mu E_\nu - \varepsilon_{\mu\nu\lambda\beta} E_\lambda E_\beta - \delta_{\mu\nu} E_\lambda E_\lambda \quad (104)$$

where  $\varepsilon_{\mu\nu\lambda\beta}$  is the four-dimensional permutation symbol and the  $E_\mu$  are the components of the electric field four-vector. The set of components for the energy-momentum tensor  $T_{\mu\nu}$  in (103) is only one of the possible sets of components for  $\mathbf{T}$ . By choosing alternate sets of rules for the products of basis vectors for  $(\mathbf{dE})\mathbf{E}$  in (102), we will obtain different sets of components for  $\mathbf{T}$ .

If we consider all possible representations of  $T_{\mu\nu}$ , of which (103) is only one, all of the components of  $\mathbf{T}$  vanish, and we can say that

$$\mathbf{T} = 0 \quad (105)$$

We also note that, in this case, we may write

$$T_{\mu\nu,\nu} = 0 \quad (106)$$

We find that each term in  $T_{\mu\nu}$  contains contributions from the time component  $E_t$  as well as every other component of the electric field four-vector, so that *all* of the fields are represented in every component of  $T_{\mu\nu}$ .

It is important to remember that the  $E_\mu$  in (103) and (104) are the components of the *generalized* electric field, thus they contain the components of the conventional electric, magnetic, and we suspect, the gravitational and nuclear fields as well. Note also, that the indices of the components of (103) and (104) range from 1 to 4, not 1 to 3, so that the energy-momentum tensor (103) has the same form, throughout, unlike the conventional electromagnetic energy-momentum tensor.

In order to simplify (104), we introduce the tensor  $E_{\mu\nu}$  with components

$$E_{\mu\nu} = 2E_\mu E_\nu - \varepsilon_{\mu\nu\lambda\beta} E_\lambda E_\beta - \delta_{\mu\nu} E_\lambda E_\lambda \quad (107)$$

Inserting (107) into (104), we get

$$E_{\mu\nu} = 8\pi T_{\mu\nu} \quad (108)$$

## 14 Mass

We intend to show, now, that mass can be expressed, entirely, as a function of charge  $q$  and static electric potential  $\phi$ . From (66) and (78) and remembering from (54) that  $U_\nu U_\nu = c^2$ , we can write

$$J_\nu A_\nu = \frac{\rho U_\nu}{c} \frac{\phi U_\nu}{c} = \rho\phi \frac{U_\nu U_\nu}{c^2} = \rho\phi \quad (109)$$

Since  $\rho$  and  $\phi$  are both scalar invariants, the product of the two is also a scalar invariant which we will call  $\Phi$ , where  $\Phi = \rho\phi$ . Inserting this and (109) into (101), we have

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{\partial\Phi}{\partial x_\mu} \quad (110)$$

Taking the inner product of the force density four-vector  $\mathbf{F}$  and the four-velocity  $\mathbf{U}$  in (110), we get

$$U_\mu F_\mu = U_\mu \frac{dp_\mu}{d\tau} = \frac{\partial\Phi}{\partial x_\mu} \frac{dx_\mu}{d\tau} \quad (111)$$

The momentum density (99) in component form is  $p_\mu = \rho_m U_\mu$ , so the second term in (111) is

$$U_\mu \frac{dp_\mu}{d\tau} = U_\mu \left( U_\mu \frac{d\rho_m}{d\tau} + \rho_m \frac{dU_\mu}{d\tau} \right) \quad (112)$$

Now, since

$$U_\mu \frac{dU_\mu}{d\tau} = \frac{1}{2} \frac{d}{d\tau} (U_\mu U_\mu) = \frac{1}{2} \frac{d}{d\tau} (c^2) = 0 \quad (113)$$

we can write (112) as

$$U_\mu \frac{dp_\mu}{d\tau} = U_\mu U_\mu \frac{d\rho_m}{d\tau} = c^2 \frac{d\rho_m}{d\tau} \quad (114)$$

Due to the chain rule

$$\frac{d\Phi}{d\tau} = \frac{\partial\Phi}{\partial x_\mu} \frac{dx_\mu}{d\tau} \quad (115)$$

and (114), we can write the last two equalities in (111) as

$$c^2 \frac{d\rho_m}{d\tau} = \frac{d\Phi}{d\tau} \quad (116)$$

In the trivial case  $\rho_m = \text{constant}$ , we have  $U_\mu F_\mu = 0$ , otherwise (116) becomes

$$c^2 \rho_m = \Phi + k_1 \quad (117)$$

where  $k_1$  is a constant. Now  $\rho_m = m_0/v_0$ , where  $m_0$  is the proper mass of the distribution of charges and  $v_0$  is the proper volume containing the charges. If we insert  $\Phi = \rho\phi$  back into (117) and remember that  $\rho = q_0/v_0$ , where  $q_0$  is the proper charge of the distribution of charges, we get

$$m_0 c^2 = q_0 \phi + k_2 \quad (118)$$

where  $k_2 = v_0 k_1$ . If  $k_2 = 0$ , then (118) becomes

$$m_0 = \frac{q_0 \phi}{c^2} \quad (119)$$

or, in relativistic form,

$$m = \frac{q\phi}{c^2} \quad (120)$$

where  $m = m_0 \sqrt{1 - U^2/c^2}$  and  $q = q_0 \sqrt{1 - U^2/c^2}$ . Notice that, at  $U = c$ , both  $m$  and  $q$  are zero.

## 15 Covariant Formulation in General Coordinates

Until now, we have limited our study to four-dimensional rectangular coordinates in Euclidean spacetime. However, in order to describe our laws in general four-dimensional coordinate systems in Euclidean spacetime one must make several adjustments:

(i) In order to handle more general coordinate systems, every previous occurrence of the Kronecker delta  $\delta_{\mu\nu}$  must be replaced by the metric tensor,  $g_{\mu\nu}$ .

(ii) Where previously it was unnecessary to distinguish between covariant and contravariant components of tensors, since there is no distinction between the two in rectangular coordinates, we must specify which we are using.

(iii) In rectangular coordinates, the derivative of a tensor is, simply, the ordinary derivative. But in general coordinates, we have terms which may include nonzero Christoffel symbols. These terms vanish in rectangular coordinates since the components of the metric tensor are constants. In general coordinates, however, the components of the metric tensor are not always constant, thus, the Christoffel symbols do not vanish, in general. Because of this, all ordinary derivatives of tensors must be replaced by their covariant derivatives.

## 16 Conclusions

The special Lorentz transformation describes an imaginary rotation of coordinate systems. We have described a *real* rotation of coordinate systems including an *automatic* rotation transverse

to the direction of motion, which the Lorentz transformations do not describe. These rotations are not necessarily observable as rotations, however, but as precession, time dilation, length contraction, etc.

In conventional theory, the magnitude of a charged particle is invariant. In this paper, the magnitude of a charge is reduced as its velocity is increased, however, the charge *density* in the region of the charge is invariant. Thus, the quantity of charge in a given volume does not change as long as no charge enters or leaves the region. It is the invariance of charge density, not the conservation of charge, which accounts for the neutrality of atoms. This indicates that charge is most likely a property of the field, rather than the particle itself.

It can be seen from the  $T_{44}$  component of (103) that the energy of the field can be negative. Rather than interpreting this as a liability, we suggest the possibility that the field of a particle *is* its antiparticle. We suspect that the creation of a particle travelling forward in time is accompanied by the creation of its antiparticle travelling backward in time.<sup>2</sup> Since an antiparticle travelling backward in time may be said to have negative energy and since the energy of the field from  $T_{44}$  is negative for  $E_t^2 < E_x^2 + E_y^2 + E_z^2$ , we suggest that the antiparticle travelling backward in time appears to us as the field of the particle. We are *not* referring, for example, to the particle/antiparticle pair created in the disintegration of an energetic photon. In that case, there is an electron and an anti-electron (positron) created. However, both of these “particles” are travelling *forward* in time. Associated with each of these particles, is a field which we claim is its antiparticle travelling *backward* in time. These particle/field (or particle/antiparticle) pairs are the pairs to which we refer. In this case, there are actually two particle/antiparticle pairs created. The electron and its field (antiparticle) comprise one particle/antiparticle pair, and the positron and its field (antiparticle) comprise the other particle/antiparticle pair. Since the antiparticles appear to us as (and are) the fields of the particles, one might even entertain the notion, that space and matter travelling backward in time are one and the same thing.

Due to this apparent particle/antiparticle link, we also conclude that there are *no* electric monopoles (since every charged particle is accompanied by its field), just as there are apparently no magnetic monopoles. These conclusions might offer a logical explanation for the puzzling absence of antimatter in the universe. If our suspicions are correct, this “missing” antimatter exists all around us as the fields of matter and, possibly, as space itself.

The equation for an expanding four-dimensional “spherical” light wave front is (8), in this paper. The wave front propagates in *four-dimensions* (3 space and 1 time), rather than the usual three-dimensions (3 space), as in conventional theory. This assures that the spatial speed of light in all inertial reference frames is the invariant  $c$ .

The form and terminology of many of the equations in this paper are, deceptively, similar to those of conventional physics, however, they differ in several ways. It is important that one not assume the equivalence of the definitions presented here with the analogous definitions in conventional theory. In most cases, they are not exactly the same. For example, the definition of the scalar electric potential  $\phi$  includes the spacetime interval  $s$  in the denominator, *not* the spatial interval  $r$ . This prevents an infinite potential at  $r = 0$ , a problem that plagues conventional electromagnetic theory.

It would also be incorrect to assume that the electric field is defined in the same manner as in

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<sup>2</sup>We refer to particles (or antiparticles) travelling forward in time as particles, since they have the characteristics of particles, and particles (or antiparticles) travelling backward in time as antiparticles having the characteristics of fields.

conventional electromagnetic theory. For example, it is easy to interpret the components of the energy-momentum tensor  $\mathbf{T}$  incorrectly as containing only the components of the conventional electric field when, in fact, they contain the components of the electric, magnetic, and possibly the gravitational and nuclear fields, as well.

We have provided, here, what we consider to be a foundation upon which a deeper and more unified understanding of laws of nature might be built.