

# Energy Density Correction

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<http://www.softcom.net/users/der555/enerdens.pdf>

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The energy density  $u$  of a continuous charge distribution, according to popular belief, in SI units, is

$$u = \frac{\epsilon_0}{2} E^2 \quad (1)$$

where  $\epsilon_0$  is the permittivity constant and  $E$  is the conventional electric field strength. I intend to show, using conventional terminology, that this should instead be

$$u = \epsilon_0 E^2 \quad (2)$$

In order to simplify, I'll start with the energy of a point charge distribution. The energy of the distribution is just the work required to assemble the distribution. Let's start with a pair of identical charged particles,  $q_1$  and  $q_2$ , at  $+\infty$  and  $-\infty$ , respectively. The work  $W_{12}$  required to bring  $q_1$  in from  $+\infty$  to the origin against the field of  $q_2$ , is

$$W_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} \quad (3)$$

where  $(1/4\pi\epsilon_0)(q_2/r_{12})$  is the potential at the location of  $q_1$  due to  $q_2$ ,  $r_{12}$  is the distance between  $q_1$  and  $q_2$  after we're through bringing in  $q_1$ . But, since  $q_2$  is not in the picture yet ( $r_{12} = \infty$ ), the work required to bring  $q_1$  in is zero, i.e.,  $W_{12} = 0$ .

The work  $W_{21}$  necessary to bring in  $q_2$  is

$$W_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_2 q_1}{r_{21}} \quad (4)$$

However, as we bring in  $q_2$ , we must do work on  $q_1$  against the field of  $q_2$  in order to keep it in place at the origin.<sup>1</sup> The work necessary to keep  $q_1$  in place, is the same as the work that would be required to bring  $q_1$  in against the field of  $q_2$ , had we brought  $q_2$  in first. But this is the same

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<sup>1</sup>This contradicts the conventional definition of work in which work can only be done on a body that undergoes a displacement, however, I believe this definition is incomplete. Please refer to Section 19 in <http://www.softcom.net/users/der555/newtransform.pdf> for my definition of work density.

as (3) for  $r_{12} = r_{21}$ . So the total magnitude of the work  $W$  required to assemble the two particles is

$$W = W_{12} + W_{21} = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1 q_2}{r_{12}} + \frac{q_2 q_1}{r_{21}} \right) \quad (5)$$

or

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^2 \sum_{j=1}^2 \frac{q_i q_j}{r_{ij}} \quad (6)$$

for  $i \neq j$ .

For three particles, we have to add the work required to bring in a third particle  $q_3$  from infinity against the fields of both  $q_1$  and  $q_2$

$$W_{31} + W_{32} = \frac{1}{4\pi\epsilon_0} \left( \frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}} \right) \quad (7)$$

to the work required to assemble  $q_1$  and  $q_2$  from (5). But again we have to include the work required to keep  $q_1$  and  $q_2$  in position as we bring in  $q_3$ . The work  $W_{13}$  and  $W_{23}$  required to keep  $q_1$  and  $q_2$  in position is

$$W_{13} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r_{13}} \quad (8)$$

and

$$W_{23} = \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}} \quad (9)$$

respectively. So the total work  $W$  required to assemble the three particles, from (5), (7), (8) and (9), is

$$W = W_{12} + W_{13} + W_{21} + W_{23} + W_{31} + W_{32} \quad (10)$$

or

$$W = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_1}{r_{21}} + \frac{q_2 q_3}{r_{23}} + \frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}} \right) \quad (11)$$

which we can write as

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^3 \sum_{j=1}^3 \frac{q_i q_j}{r_{ij}} \quad (12)$$

for  $i \neq j$ . So for any number of particles  $n$ , we can apparently write

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \sum_{j=1}^n \frac{q_i q_j}{r_{ij}} \quad (13)$$

for  $i \neq j$ . We can also write this as

$$W = \sum_{i=1}^n q_i \left( \sum_{j=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) \quad (14)$$

for  $i \neq j$ . The term in brackets is the potential  $\phi(\mathbf{r}_i)$  at the point  $\mathbf{r}_i$  (the position of  $q_i$ ) due to all other charges. Thus, we can write (14) as

$$W = \sum_{i=1}^n q_i \phi(\mathbf{r}_i) \quad (15)$$

For a volume charge density  $\rho$ , (15) becomes

$$W = \int \rho\phi dV \quad (16)$$

where  $dV$  is the volume element. Using Poisson's equation  $\nabla^2\phi = -\rho/\epsilon_0$  we can write (16) as

$$W = -\epsilon_0 \int \phi \nabla^2\phi dV \quad (17)$$

Integration by parts then leads to

$$W = \epsilon_0 \int |\nabla\phi|^2 dV \quad (18)$$

and, since  $E = |\nabla\phi|$ , we get

$$W = \epsilon_0 \int E^2 dV \quad (19)$$

This represents the energy stored in the electric field, therefore, we can interpret  $\epsilon_0 E^2$  as the energy density  $u$ , or

$$u = \epsilon_0 E^2 \quad (20)$$

Clearly, this is in contradiction to the conventional physics claim that the energy density is  $u = (\epsilon_0/2)E^2$ . The extra energy density is due to the extra work that must be done on the charges already assembled in order to keep them in place against the field of each new charge as we add it to the configuration.