

Biot-Savart's Companion

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E-mail: drutherford@softcom.net
<http://www.softcom.net/users/der555/biotcomp.pdf>

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1 The Companion Law

The Biot-Savart law states that the element of magnetic field $d\mathbf{B}$ produced by a short segment $d\mathbf{l}$ of wire of arbitrary shape carrying a steady line current I , in SI units, is

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \quad (1)$$

with magnitude

$$|d\mathbf{B}| = dB = \frac{\mu_0}{4\pi} \frac{I dl \sin \theta}{r^2} \quad (2)$$

where θ is the angle between $d\mathbf{l}$ and $\hat{\mathbf{r}}$.

I believe the Biot-Savart law has a companion law that, to my knowledge, has gone undiscovered until now. My companion laws to (1) and (2) introduce a 'scalar' field H whose element dH has similar form to (1) and (2).¹

The companion law is derived from $\nabla \cdot \mathbf{A}$, where \mathbf{A} is the vector potential due to a 'point' charge q' moving with constant velocity \mathbf{v}' .² We first note that $\mathbf{A} = \mathbf{v}'\phi/c^2$, where $\phi = q'/(4\pi\epsilon_0 r)$ is the scalar electric potential, due to q' , at a field point P a distance r from q' .³ Thus, after substitution we get

$$\nabla \cdot \mathbf{A} = \nabla \cdot \left(\frac{\mathbf{v}'\phi}{c^2} \right) = -\frac{q'}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \quad (3)$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from q' to P .

If we refer to $\nabla \cdot \mathbf{A}$ here as H , and note that $\epsilon_0\mu_0 = 1/c^2$, we can write (3) as

$$H = -\frac{\mu_0 q'}{4\pi} \frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \quad (4)$$

¹See <http://www.softcom.net/users/der555/newtransform.pdf>. The field H is actually part of my generalized electric field (however, it is not referred to as H , there). Note that, here, I am using a three-dimensional, non-relativistic treatment, as opposed to the four-dimensional, relativistic treatment used at the above link.

² $\nabla \cdot \mathbf{A}$ is physical and, in general, nonzero in my theory.

³It is important to note that the scalar potential ϕ , here, is the three-dimensional part of my four-dimensional potential at <http://www.softcom.net/users/der555/newtransform.pdf>, since this is a three-dimensional treatment.

Since $\mathbf{v}' \cdot \hat{\mathbf{r}}$ is a scalar quantity (in three-dimensional space), we could also write (4) as

$$H = -\frac{\mu_0 q'}{4\pi} \frac{v' \cos \theta}{r^2} \quad (5)$$

where v' is the magnitude of \mathbf{v}' and θ is the angle between \mathbf{v}' and $\hat{\mathbf{r}}$.

The element dH at P due to a ‘point’ charge dq' is

$$dH = -\frac{\mu_0 dq'}{4\pi} \frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \quad (6)$$

or

$$dH = -\frac{\mu_0 dq'}{4\pi} \frac{v' \cos \theta}{r^2} \quad (7)$$

Now consider a small segment $d\mathbf{l}$ of wire carrying a steady current I , within which the ‘point’ charge dq' is moving with velocity \mathbf{v}' parallel to $d\mathbf{l}$. In terms of the current I , substituting $dq' = Idt$ and $\mathbf{v}' = d\mathbf{l}/dt$, (6) and (7) become

$$dH = -\frac{\mu_0}{4\pi} \frac{I d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (8)$$

or

$$dH = -\frac{\mu_0}{4\pi} \frac{I dl \cos \theta}{r^2} \quad (9)$$

where dl is the magnitude of $d\mathbf{l}$. The equations (8) and (9) are my companions to (1) and (2), respectively.

We can find the total field H at P by integrating (8) along the wire, so that

$$H = -\frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (10)$$

2 Force on a Test Charge in the Field H

The *additional force* \mathbf{F}_a on a test charge q at point P moving with velocity \mathbf{v} in the field H at P is⁴

$$\mathbf{F}_a = -q\mathbf{v}H \quad (11)$$

If H at P is due to a ‘point’ charge q' moving with constant velocity \mathbf{v}' , we can substitute (4) or (5) into (11), to obtain

$$\mathbf{F}_a = -q\mathbf{v} \left(-\frac{\mu_0 q'}{4\pi} \frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \right) = \frac{\mu_0 qq'}{4\pi} \frac{\mathbf{v}(\mathbf{v}' \cdot \hat{\mathbf{r}})}{r^2} \quad (12)$$

or

$$\mathbf{F}_a = \frac{\mu_0 qq'}{4\pi} \frac{\mathbf{v}(v' \cos \theta)}{r^2} \quad (13)$$

respectively.

⁴See <http://www.softcom.net/users/der555/actreact.pdf>.

For a steady line current I , the element of force $d\mathbf{F}_a$ on q due to a short segment $d\mathbf{l}$ of wire, using (8), is

$$d\mathbf{F}_a = -q\mathbf{v}dH = \frac{\mu_0 q\mathbf{v}}{4\pi} \frac{I d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (14)$$

The total force \mathbf{F}_a on q can be found by integrating (14) along the wire, resulting in

$$\mathbf{F}_a = \frac{\mu_0 q I \mathbf{v}}{4\pi} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (15)$$

It is interesting to note that the magnetic force $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$ changes the *direction* of the velocity of a test particle, but not the magnitude of its velocity. In contrast, my additional force $\mathbf{F}_a = -q\mathbf{v}H$ changes the *magnitude* of the velocity, but not its direction. These two forces go hand-in-hand, along with the electric force $\mathbf{F}_e = q\mathbf{E}$, to complete the three-dimensional, non-relativistic equations of motion of a test particle.⁵

These equations predict new physics, but the most interesting prediction involves the time component of the force. Analyzing the time component of the force is outside of the three-dimensional treatment here, however due to its implications I would like to mention it.

The time component of the force on q , due to q' , is⁶

$$F_t = q \left(\frac{1}{c} \mathbf{v} \cdot \mathbf{E} + c \nabla \cdot \mathbf{A} \right) \quad (16)$$

The last term is the time component of the additional force

$$F_{ta} = qc \nabla \cdot \mathbf{A} \quad (17)$$

or

$$F_{ta} = qcH \quad (18)$$

Substituting (4) into (18), we obtain

$$F_{ta} = -\frac{\mu_0 q q' c}{4\pi} \frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \quad (19)$$

For a steady line current I , using the same methods as above, we find that the element of force dF_{ta} on q due to a short segment $d\mathbf{l}$ of wire, is

$$dF_{ta} = -\frac{\mu_0 qc}{4\pi} \frac{I d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (20)$$

The total time component of the additional force F_{ta} on q can be found by integrating (20) along the wire, to obtain

$$F_{ta} = -\frac{\mu_0 qc I}{4\pi} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (21)$$

If we extend Newton's Second Law $\mathbf{F} = m\mathbf{a}$ to include the time component of the force F_t , that is $F_t = ma_t$, where a_t is the time component of the acceleration, then we can write the time component of the additional force as $F_{ta} = ma_{ta}$, where a_{ta} is the acceleration due to the

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time component of the additional force F_{ta} .⁷ Thus we can now write the time component of the acceleration a_{ta} of q , due to F_{ta} , as

$$a_{ta} = -\frac{\mu_0 qcI}{4\pi m} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \quad (22)$$

where m is the mass of q .

Note that c in F_{ta} and dF_{ta} is the time component of the velocity of q , thus q is *stationary* in space.⁸ Also note that, for a nonzero F_{ta} (or dF_{ta}), there is a *nonzero* acceleration in time. In other words, (22) describes a means of transporting q in time.

⁷There are terms in addition to the time rate of change of four-momentum in my four-dimensional, relativistic treatment at <http://www.softcom.net/users/der555/newtransform.pdf>. These terms are considered to be relativistic, however, and are not considered here.

⁸I'm focusing, here, on the case where q is stationary in F_t . There might also be an acceleration in time due to $(q/c) \mathbf{v} \cdot \mathbf{E}$ if q is not stationary in space.